1) Denote by \([x]\) the integer part of \(x\). Prove that for any natural number \(n\), the integer \(\lfloor (2 + \sqrt{3})^n \rfloor\) is odd.

2) Prove that for every \(b\), in the numeral system of base \(b\), none of the numbers

\[10101, \ 1010101, \ 101010101, \ldots\]

is a prime number.

3) Let \(f(x)\) be a continuous positive 1-periodic function on \(\mathbb{R}\), and let \(\alpha\) be a real number. Prove that

\[\int_0^1 \frac{f(x)}{f(x + \alpha)} \, dx \geq 1.\]

4) The Fibonacci sequence 1, 1, 2, 3, 5, 8, \ldots is defined by the linear recursive rule \(F_{n+2} = F_{n+1} + F_n\) \((n \geq 0)\), with the initial values \(F_0 = F_1 = 1\). Prove that for any integer \(n > 0\), there are infinitely many terms of this sequence divisible by \(n\).

5) Let \(n\) be a natural number, which is coprime with 10. Prove that \(n\) has infinitely many multiples whose digits are all ones.

6) Let \(\{a_n\}, \{b_n\}\) two sequences of real numbers that satisfy

\[\limsup_{n \to \infty} a_n = \limsup_{n \to \infty} b_n = +\infty.\]

Prove that there exist indices \(m\) and \(n\) such that \(|a_m - a_n| > 1\) and \(|b_m - b_n| > 1\).