1) Let
\[
\frac{N}{M} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1350} + \frac{1}{1351},
\]
where \(N, M\) are natural numbers. Prove that the numerator \(N\) is a multiple of 2027.

2) Prove that the diophantine equation
\[x^{n-1} + 1 = y^n\]
(where \(n\) is an integer, \(n \geq 3\)) has no positive solutions \(x, y \in \mathbb{N}\) such that \(n\) and \(x\) are relatively coprime.

3) Given a point \(p\) in \(\mathbb{R}^3\), a plane \(\pi\) passing through \(p\), and some \(r > 0\), we define the circle of centre \(p\) and radius \(r\) contained in \(\pi\) as the set of all points of \(\pi\) whose distance to \(p\) is \(r\). Prove that \(\mathbb{R}^3\) is a union of disjoint circles.

4) Let \(n \geq 0\). Show that there is a constant \(C\) depending only on \(n\) such that for every monic polynomial \(P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0\), every \(b \in \mathbb{R}\) and \(u > 0\) the following inequality holds:
\[
\int_b^{b+u} |P(x)| \, dx \geq Cu^{n+1}.
\]

5) The sequence \(\{x_n\}_{n=1}^\infty\) of positive real numbers decreases and satisfies \(\sum_{n=1}^\infty x_n = \infty\). Prove that
\[
\sum_{n=1}^\infty x_n \exp\left(-\frac{x_n}{x_{n+1}}\right) = \infty.
\]

6) For \(n \geq 1\), let \(P_n(x) = x(x-1)(x-2)\cdots(x-n+1)\), and let \(P_0(x) = 1\). Prove that
\[
P_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} P_k(x) P_{n-k}(y),
\]
for every \(n \geq 0\), and every \(x, y \in \mathbb{R}\).

7) Every point in the plane is colored either red or blue. Show that one can always find an equilateral triangle, whose three vertices are of the same color.