1) Let \( n \) be an odd integer, \( n \geq 3 \). Let \( S \) be the set of integers \( x \) such that \( 1 \leq x \leq n \), and \( x \) and \( x + 1 \) are both coprime with \( n \). Show that

\[
\prod_{x \in S} x \equiv 1 \mod n.
\]

**Solution:** Let \( x \in S \). Since \( x \) and \( n \) are coprime, there is a unique \( 1 \leq y \leq n \) such that \( xy \equiv 1 \mod n \). Let us check that \( y \in S \) also. Clearly \( y \) is coprime with \( n \). Since \( x(y+1) \equiv x+1 \mod n \), and \( x+1 \) is coprime with \( n \), we see that \( y+1 \) is also coprime with \( n \). Therefore, \( y \in S \).

Suppose that \( x = y \). Then \( 0 \equiv x^2 - 1 = (x+1)(x-1) \mod n \). Since \( x+1 \) is coprime with \( n \), we get \( x-1 \equiv 0 \mod n \), hence, \( x = 1 \). It follows that the elements of \( S \setminus \{1\} \) can be arranged in pairs in such a way that the product of each pair is 1. This solves the problem.

2) Show that for any natural numbers \( n \) and \( k \), the number \( k!(n!)^k \) always divides \( (kn)! \).

**Solution:** We will give a combinatorial argument. Assume that one has \( kn \) distinct objects. We choose \( k \) different colours, and for each object we paint it with one of these colours and write a number from \( \{1, \ldots, n\} \) on it, in such a way that no two different objects have both the same number and colour. Since there are \( kn \) objects, we get \( (kn)! \) possible assignments of this kind. Note that in each of these assignments, each of the \( k \) colours appears exactly \( n \) times.

Now let us define an equivalence relation on the set of these \( (kn)! \) assignments. We say that two assignments are equivalent if when looking only at the colours, one can convert one of the assignments into the other one just by permuting the colours.

Let us count the number of assignments in an equivalence class. Given any assignment, one can pass to an equivalent assignment by first doing a permutation of the \( k \) colours and then, for each of the \( k \) colours, permuting the \( n \) objects that are painted with this colour. Since these steps are independent and we get different assignments if we do the permutations in a different way, we see that any equivalence class contains \( k!(n!)^k \) assignments (\( k! \) is the number of permutations of \( k \) colours and each \( n! \) amounts to the permutation of objects that correspond to one of the \( k \) colours).

Since all equivalence classes have the same number of elements, we get that \( (kn)! = pk!(n!)^k \), where \( p \) is the number of equivalence classes. Hence, \( k!(n!)^k \) divides \( (kn)! \).

3) Find all real numbers \( a \) such that \( f(x) = \{ax + \sin x\} \) is periodic. Here \( \{y\} \) denotes the fractional part of \( y \).

**Solution:** Let \( T \) be a period of \( f \). Then \( a(x+T) + \sin(x+T) - ax - \sin x \in \mathbb{Z} \) for any \( x \in \mathbb{R} \). By continuity, \( \sin(x+T) - \sin x \) is a constant not depending on \( x \), which only can be zero (if not, at a point \( x \) where \( \sin(x+T) = \pm 1 \) we would have \( |\sin x| > 1 \)). Hence \( T = 2k\pi \), where \( k \) is an integer. It follows that \( a = \frac{q}{\pi} \), where \( q \in \mathbb{Q} \).

Conversely, for any \( a \) of this form, \( f \) is periodic.

4) Suppose that \( P \) is a polynomial with integer coefficients such that for any natural \( n \), \( P(n) \) is of the form \( 2^k \) for some natural number \( k \). Prove that \( P \equiv \text{const} \).

**Solution:** Notice that \( P(x) \) cannot tend to \(-\infty \) when \( x \) goes to \(+\infty \). Suppose that the polynomial \( P \) is not constant. Then its degree is at least one and its main coefficient is positive. Since \( P' \) has a finite number of roots, there is some \( N_0 \in \mathbb{N} \) such that \( P(x) \) strictly increases for \( x \geq N_0 \). Put \( P(N_0) = 2^{k_0} \), then \( P(n+1) > P(n) \) for all \( n \geq N_0 \), and it follows that \( P(n) \geq 2^{k_0+n-N_0}, n \geq N_0 \). However, such growth at \(+\infty \) is impossible for a polynomial, a contradiction. This shows that \( P \equiv \text{const} \).