1) Under what conditions on the eigenvalues of a complex matrix $T$ does there exist a polynomial $P$ with positive coefficients such that $P(T) = 0$?

2) Show that for any natural number $n$, there exist infinitely many Fibonacci numbers divisible by $n$.

3) Let $\{r_n\}_{n=1}^\infty$ be fixed non-negative real numbers such that $\sum_{n=1}^{\infty} r_n < \infty$. If $\{\theta_n\}_{n=1}^\infty$ is a sequence of real numbers, let $z(\{\theta_n\})$ be the complex number defined by

$$z(\{\theta_n\}) = \sum_{n=1}^{\infty} r_n e^{i\theta_n}.$$

Describe the set $X$ of numbers $z(\{\theta_n\})$ that can be obtained by choosing sequences $\{\theta_n\}_{n=1}^\infty$ of real numbers, i.e., the set

$$X = \{z(\{\theta_n\}) : \theta_n \in \mathbb{R}, n = 1, 2, \ldots\}.$$

4) Prove that the following numbers (where $m$ and $n$ are natural numbers) cannot be integers:

1. $M = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$;
2. $N = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+m}$;
3. $K = \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1}$.

5) Prove that the only solution in integers of the equation

$$x^2 + y^2 + z^2 = 2xyz$$

is $x = y = z = 0$. 