1. Given $f : [0,1] \rightarrow \mathbb{R}$ with $f \in L^2([0,1])$, extend $f$ to $\mathbb{R}$ to obtain a function $\tilde{f}$ odd with respect to the origin, even with respect to 1 and -1 and 4-periodic. By writing the Fourier series of $\tilde{f}$ in $[-1,1]$ in terms of sines and cosines show that the cosine coefficients are zero as well as the even sine coefficients. Prove that

$$\{\sqrt{2}\sin\left(\frac{2k+1}{2}\pi x\right) : k = 0, 1, 2, \ldots \}$$

is and orthonormal basis of $L^2([0,1])$. This is called the \textbf{sine-IV} basis for $L^2([0,1])$.

2. Show that for a function $f \in L^2(\mathbb{R}) \cap C^2(\mathbb{R})$, the coefficients of $f$ in the block cosine-I basis given by

$$\{\chi_{[n,n+1)}(x) : n \in \mathbb{Z}\} \cup \{\chi_{[n,n+1)}(x)\sqrt{2}\cos\pi k(x-n) : n \in \mathbb{Z}, k = 1, 2, \ldots \}$$

decay, for $n$ fixed, at a rate proportional at least to $1/k^2$.

3. Given $\varepsilon > 0$, choose $\psi$ an even, $C^\infty$ function defined on $\mathbb{R}$, supported on $[-\varepsilon, \varepsilon]$ such that $\int_{-\varepsilon}^{\varepsilon} \psi(x) = \pi/2$. Let $\theta(x) = \int_{-\infty}^{x} \psi(y)dy$. Show that $\theta(x) + \theta(-x) = \pi/2$. Define $s_\varepsilon(x) = \sin \theta(x)$. Show that $[s_\varepsilon(x)]^2 + [s_\varepsilon(-x)]^2 = 1$.

4. With the same notation as in the previous exercise, let $c_\varepsilon(x) = \cos(\theta(x))$. Let $I = [\alpha, \beta] \subset \mathbb{R}, \varepsilon, \varepsilon' > 0$, such that $\alpha + \varepsilon < \beta - \varepsilon'$. The function

$$b_I(x) = s_\varepsilon(x-\alpha)c_{\varepsilon'}(x-\beta)$$

is called a \textbf{bell} function associated with the interval $I = [\alpha, \beta]$.

a) Sketch the graph of the bell function $b_I$.

b) Show that on $[\alpha - \varepsilon, \alpha + \varepsilon]$

i) $b_I(x) = s_\varepsilon(x-\alpha)$.

ii) $b_I(2\alpha - x) = s_\varepsilon(\alpha - x) = c_\varepsilon(x-\alpha)$.

iii) $b_I^2(x) + b_I^2(2\alpha - x) = 1$.

5. Show that the collection of $N$ vectors

$$\mu_k \frac{1}{\sqrt{N}} \left( \sin \frac{k\pi}{N}(n + \frac{1}{2}) \right)_{n=-N}^{N-1}, \quad k = 1, 2, \ldots, N,$$

each one of size $2N$, where $\mu_k = 1$ if $k = 1, 2, \ldots, N - 1$ and $\mu_N = 1/\sqrt{2}$, is an orthonormal system.
6. Show that the $N$ vectors given by
\[ \sqrt{\frac{2}{N-1}} \left( \lambda_n \cos \left( \frac{\pi}{N-1} kn \right) \right)_{n=0}^{N-1}, \quad k = 0, 1, 2, \ldots, N-1, \]
each one of size $N$, where $\lambda_0 = 1/\sqrt{2}$, $\lambda_{N-1} = 1/\sqrt{2}$ and $\lambda_n = 1$ if $n = 1, 2, \ldots, N-2$, is an orthonormal basis of the space of signals of size $N$. This basis corresponds to the one obtained by extending $f = (f(n))_{n=0}^{N-1}$ evenly with respect to $n = 0$.

7. (2 points) This exercise shows how to calculate DCT-I with an induction relation that involves DCT-IV.
   a) Regroup the terms $f(n)$ and $f(N-1-n)$, $0 \leq n \leq \frac{N}{2} - 1$, $N = 2^q$, in the DCT-I, to write $\hat{f}_I(2k)$ as the DCT-I of the signal
   \[ s(n) = \frac{1}{\sqrt{2}} [f(n) + f(N-1-n)], \quad 0 \leq n \leq \frac{N}{2} - 1. \]
   b) With the same technique as in part a), write $\hat{f}_I(2k + 1)$ as the DCT-IV of the signal
   \[ r(n) = \frac{1}{\sqrt{2}} [f(n) - f(N-1-n)], \quad 0 \leq n \leq \frac{N}{2} - 1. \]
   c) Using that, with a fast algorithm, the number of operations to calculate DCT-IV of size $N$ is $O(N \log_2 N)$ and parts a) and b), show that with the above algorithm, the number of operations needed to calculate DCT-I of size $N$ is also $O(N \log_2 N)$.

8. Show that the number $B_j^{(2)}$ of orthogonal bases of the space of discrete images of size $N^2(N = 2^L)$ in a bi-dyadic tree of depth $j$, $0 \leq j \leq L$, satisfies
   \[ 2^{4j-1} \leq B_j^{(2)} \leq 2^4 4^{j-1}. \]

9. Consider the signal $f$ of size $N = 8$ given by
   \[ f = (8, 16, 24, 32, 40, 48, 56, 64). \]
   a) Compute the DCT-I of $f$, rounding the result to the nearest integer. Compress the signal 50\% by setting to zero the DCT-I coefficients in positions 4, 5, 6, and 7. Find now the inverse DCT-I of this compressed signal, and, after rounding, observe that is similar to the original one.
   b) Take now the orthonormal basis of $\mathbb{C}^8$ given by
   \[ \left\{ \lambda_k \frac{1}{2} \left( \cos \frac{\pi kn}{4} \right)_{n=0}^{7} \right\}_{k=0}^{4} \bigcup \left\{ \frac{1}{2} \left( \sin \frac{\pi kn}{4} \right)_{n=0}^{7} \right\}_{k=1}^{3}, \]
where $\lambda_0 = \lambda_4 = \frac{1}{\sqrt{2}}$ and $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Repeat the process in a), setting now to zero the frequencies $k = 3$ and $k = 4$ of cosines, and the frequencies $k = 2$ and $k = 3$ of sines. Observe that the final result is somehow different than the original signal.