1. Show that if \( f(x) = \frac{1}{T} \chi_{[-T/2, T/2]}(x), \ x \in \mathbb{R}, \) then
\[
\mathcal{F}(f)(\xi) = \frac{\sin T\pi \xi}{T\pi}, \ \xi \in \mathbb{R}.
\]
(The function \( h(t) = \frac{\sin \pi \xi}{\pi \xi} \) is called the sinc (sinus cardinalis) function and plays an important rôle in signal processing.)

2. Let \( f(x) = e^{-4\pi^2 x^2}, \ x \in \mathbb{R}. \) Show that
\[
\mathcal{F}(f)(\xi) = \frac{1}{2\sqrt{\pi}} e^{-\xi^2/4}, \ \xi \in \mathbb{R}.
\]

3. Show that if \( \varphi \in \mathcal{M}, \) the mapping \( U \) given by
\[
U \varphi(x, \xi) = e^{-2\pi i x \xi} \varphi(x, \xi),
\]
belongs to \( \tilde{\mathcal{M}}. \) Moreover, show that \( U^* \tilde{\varphi}(x, \xi) = e^{2\pi i x \xi} \tilde{\varphi}(x, \xi), \) where \( U^* \) denotes the adjoint to \( U. \) (The spaces \( \mathcal{M} \) and \( \tilde{\mathcal{M}} \) have been defined in class.)

4. Let \( V_T \) be the space of functions in \( L^1(\mathbb{R}) \) such that \( \text{supp} \mathcal{F}(f) \subset [-T/2, T/2]. \) Show that if
\[
h_T(x) = \frac{\sin(\pi T x)}{\pi T x},
\]
then \( \{h_T(x - \frac{k}{T})\}_{k=-\infty}^{\infty} \) is an orthogonal basis of \( V_T. \) If \( f \in V_T \cap L^1(\mathbb{R}) \) prove that
\[
f(\frac{k}{T}) = T \int_{-\infty}^{\infty} f(x) h_T(x - \frac{k}{T}) \, dx.
\]

5. Let \( f \in L^1(\mathbb{R}) \) and \( \text{supp} \mathcal{F}(f) \subset [-\frac{T}{2}, \frac{T}{2}]. \) Consider the function
\[
F_p(\xi) := \sum_{k=-\infty}^{\infty} \mathcal{F}(f)(\xi + Tk),
\]
which is periodic of period \( T. \) Show that, as a periodic function, the Fourier series of \( F_p \) is
\[
\sum_{n=-\infty}^{\infty} \frac{1}{T} f\left(\frac{n}{T}\right) e^{-2\pi i \frac{n}{T} \xi}.
\]

6. Given two periodic discrete signals, \( f = \{f(n)\}_{n=0}^{N-1} \) and \( h = \{h(n)\}_{n=0}^{N-1}, \) of period \( N, \) the circular convolution is defined as
\[
f \circledast h = \sum_{p=0}^{N-1} f(p) h(n-p) \quad n \in \mathbb{Z}.
\]
Prove that \( f \odot h = h \odot f \).

7. Suppose that \( \text{supp} \mathcal{F}(f) \subset \left[ -\frac{(n + 1)T}{2}, -\frac{nT}{2} \right] \cup \left[ \frac{nT}{2}, \frac{(n + 1)T}{2} \right] \). Use a similar argument to the one given in the proof of the Shannon Sampling Theorem to show that
\[
f(x) = \sum_{k=-\infty}^{\infty} f\left( \frac{k}{T} \right) \frac{\sin((n + 1)\pi(Tx - k)) - \sin(n\pi(Tx - k))}{\pi(Tx - k)}.
\]
(Observe that one can recover Shannon Sampling Theorem setting \( n = 0 \) in the above formula)

8. Denote by \( \hat{f}(k) \) de DFT of a discrete signal of size \( N \) (\( N \) even). Define \( \tilde{f}(\frac{N}{2}) = \hat{f}(\frac{3N}{2}) = \hat{f}(\frac{N}{2}) \) and
\[
\tilde{f}(k) = \begin{cases} 
2\hat{f}(k) & \text{if } 0 \leq k < \frac{N}{2} \\
0 & \text{if } \frac{N}{2} < k < \frac{3N}{2} \\
2\hat{f}(k - N) & \text{if } \frac{3N}{2} < k < 2N 
\end{cases}
\]
Prove that the discrete signal \( \tilde{f} \) of size \( 2N \) satisfies \( \tilde{f}(2n) = f(n) \).

9. Let \( f \) be the discrete signal of size 4 given by \( f = (1, 2, 3, -1) \). Compute the DFT of \( f \) using FFT. Check that your result is correct by computing DFT directly.

10. Show that the bidimensional discrete exponentials
\[
e_{k,l}(n, m) := e^{\frac{2\pi in}{N}} e^{\frac{2\pi im}{N}}, \quad 0 \leq k, \ell < N,
\]
satisfy
\[
L_g e_{k,l}(n, m) = e_{k,l}(n, m) \sum_{p=0}^{N-1} \sum_{q=0}^{N-1} g(p, q) e_{k,l}(-p, -q)
\]
where \( L_g f(n, m) = f \odot g(n, m) \), for \( g \) and \( f \) \( N \)-periodic bidimensional discrete signals.