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LECTURE 1

1.1. INTRODUCTION

1.1.1. Discovering hydrocarbon fields.

Transparency of Talk UC3M-2015-EH

To study a wave \( f(t) \), \( 0 \leq t \leq 1 \), try to write \( f \) as a
superposition of "fundamental waves"

\[
f(t) = \sum_{k=-\infty}^{\infty} c_k \cos k \omega t + \sum_{k=1}^{\infty} b_k \sin k \omega x + \sum_{k=1}^{\infty} c_k \sin k \omega t.
\]

Since \( e^{ix} = \cos x + i \sin x \) (Euler), one could try

\[
f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik \omega t}.
\]

Since \( \int_{-\infty}^{\infty} e^{ik \omega t} e^{-i \omega t} \, dt = 2 \pi \delta_{ik} \), it follows that \( C_k = \int_{0}^{1} f(t) e^{-ik \omega t} \, dt = \hat{f}(k) \).

If the wave last for \( T \) seconds, examine \( f \) in each interval \([nT, (n+1)T]\), \( 0 \leq T-1 \) the signal \( f \) \( X_{\text{in},n+1} \) to write

\[
f(t) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} c_{kn} e^{i2\pi k \omega t} \cdot X_{\text{in},n+1}(t)
\]

with \( c_{kn} = \int_{0}^{1} f(t) e^{-ik \omega t} \, dt \). More generally, one can
consider systems of the form \( f \) \( X(t) = \sum_{n=0}^{\infty} X_n(t) \), which
leads to Gabor systems to study singularities of waves.
1.1.2. Images: compression and edge detection

Introduction to the article of the Revista de la Unión Matemática Argentina - 2004

Divide $E_0, J^2$ on equal "pixels" of size $2^{-j}$ to obtain $I_{k,e}^{(i)} = \left[ \frac{k}{2^j}, \frac{k+1}{2^j} \right] \times \left[ \frac{e}{2^j}, \frac{e+1}{2^j} \right]

0 \leq k < 2^j, 0 \leq e < 2^j

If the picture is represented by $f(x,y)$ on $E_0, J^2$, a way to represent $f$ is to observe the slight subunity over each pixel,

$$p_{k,e}^{(i)} = \frac{1}{I_{k,e}^{(i)}} \int_{I_{k,e}^{(i)}} f(x,y) \, dx \, dy$$

Again, we have

$$f(x,y) \sim \sum_{k=0}^{2^j-2} \sum_{e=0}^{2^j-2} p_{k,e}^{(i)} I_{k,e}^{(i)} (x,y)$$

One could also consider expansions of the form

$$f(x,y) \sim \sum_{k=0}^{2^j-2} \sum_{e=0}^{2^j-2} p_{k,e}^{(i)} \Phi_{k,e}^{(i)} (x,y)$$

with $\Phi_{k,e}^{(i)} (x) = 2^j \phi (2^j x - k, e)$ for appropriate $\phi$. These would lead to wavelets

1) Compression: find a good approximation to $f$
   with a small number of coefficients.

2) Edge detection: position of large coefficients will
determine the edge of the image.
1.2 A Tool: The Fourier Transform in $\mathbb{R}^n$

For $f \in L^1(\mathbb{R}^n)$,

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx, \quad \xi \in \mathbb{R}^n$$

$F: L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ with $\|F\| \leq 1$ and $F$ is a continuous function by the LDC Theorem.

---

**Ex 1.1.** Show that if $f(x) = \frac{1}{T} \chi_{[-\frac{T}{2}, \frac{T}{2}]}(x)$, $x \in \mathbb{R}$, then

$$\hat{f}(\xi) = \frac{\sin \pi \xi T}{\pi \xi T} \quad (\text{sinc}(T\xi))$$

---

**Ex 1.2.** Show that if $f(x) = e^{-\frac{|x|^2}{2}}$, $x \in \mathbb{R}$, then

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}$$

---

"The Fourier transform of two different signals may be similar" (Talk UCBM - 2015)

---

$F$ defined on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ can be extended to $L^2(\mathbb{R}^n)$ and $F: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with $\|F\| = 1$, i.e., $\|Ff\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$ (Plancherel Thm). The inverse of $F$ is

$$F^{-1}g(x) = \int_{\mathbb{R}^n} g(\xi) e^{\frac{2\pi i x \cdot \xi}{\xi^2}} \, d\xi$$

Since $L^2(\mathbb{R}^n)$ is a Hilbert space, polarization gives

$$\langle Ff, Fg \rangle = \langle f, g \rangle \quad \forall f, g \in L^2(\mathbb{R}^n) \text{ (Parserval)}$$

---

Poisson Summation Formula (PSF): \[ \sum_{k \in \mathbb{Z}^n} \hat{f}(x+k) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{2\pi i x \cdot k} \text{ (PSF1)} \]

for $f \in \mathcal{S}(\mathbb{R}^n)$, and \[ \sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{-2\pi i x \cdot k} = \sum_{k \in \mathbb{Z}^n} \hat{f}(x+k) \text{ (PSF-2)} \]

---

**Ex 1.3**
1.3 SAMPLING THEOREM

\[ f(t), t \in \mathbb{R}; T > 0 \]

Sampling rate: \( \frac{1}{T} \)

Consider \( \{f(\frac{k}{T})\}_{k \in \mathbb{Z}} \)

C. Shannon (1948) and also Wiener (1936) proved that if \( \text{sup} \, \mathcal{F}f \subseteq \left[ \frac{-T}{2}, \frac{T}{2} \right] \), then \( f \) can be recovered precisely with samples \( \{f(\frac{k}{T})\}_{k \in \mathbb{Z}} \) by means of combinations of interpolations of sinc functions.

\[ \mathcal{F} \text{L}(\mathbb{R}) \text{ and } \text{sup} \, \mathcal{F}f \subseteq \left[ \frac{-T}{2}, \frac{T}{2} \right], T > 0. \text{ Then} \]

\[ f(x) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{T}\right) \frac{\sin \pi(Tx-k)}{\pi(Tx-k)} \quad \times \in \mathbb{R} \]

\( \text{Convergence in } L^2(\mathbb{R}) \).

\[ \mathcal{F}f = \mathcal{F}F \text{ where } F \text{ is entire on } \mathbb{C} \text{ of exponential type. It makes sense to consider } f\left(\frac{k}{T}\right). \]

See § 1.2 in my "Notes" or § 6.1 in [HWJ].

Uses:

- PFS-2;
- Inversion formula;
- Ex. 1.1.

Since

\[ \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{2\pi i(x-\frac{k}{T})s} ds = \frac{\sin \pi(Tx-k)}{\pi(Tx-k)} \]

\[ = \delta_t(x-\frac{k}{T}) \]

(follows from Exercise 4.1, changing

\[ s \leftrightarrow x-\frac{k}{T} \])

\[ \mathcal{F}^{-1}\left( \frac{h_T(x-k)}{\pi(Tx-k)} \right) = \frac{\sin \pi(Tx-k)}{\pi(Tx-k)} = \delta_t \left( x - \frac{k}{T} \right) \]
Take Fourier in sampling formula:

\[
\hat{Fc}(\xi) = \sum_{k=-\infty}^{\infty} c(k/T)^{2 \pi i \xi k} = \sum_{k=-\infty}^{\infty} c(k/T)^{2 \pi i \xi k} e^{-2\pi i \xi k/T}
\]

\[
= \frac{1}{T} \left( \sum_{k=-\infty}^{\infty} c(k/T) e^{-2\pi i \xi k/T} \right) \chi_{[-T/2, T/2]}(\xi)
\]

(This formula appears in the proof of the sampling theorem.)

\[
\left( \sum_{k=-\infty}^{\infty} c(k/T)^{2 \pi i \xi k} \right) \chi_{[-T/2, T/2]}(\xi) = \hat{f}(\xi)
\]

Explain aliasing phenomena when sampling rate is higher than Shannon sampling rate: \( T_0 \) s.t. \( \text{supp} \hat{f} \subset [-T_0, T_0] \) (Smallest \( T_0 \))

Sampling rate: \( T_0(f) = \text{smallest } T \text{ s.t. } \text{supp} \hat{f} \subset [-T/2, T/2] \)
Proof of Sampling Theorem

Show:

\[
\left( \sum_{k=-\infty}^{\infty} F \mathbf{f} (\xi + T k) \right) \chi_{\left[ \frac{-T}{2}, \frac{T}{2} \right]} (\xi) = F \mathbf{f} (\xi)
\] (1)

Consider \( g(x) = \frac{1}{T} \mathbf{f} \left( \frac{x}{T} \right) \):

\[
F g(\xi) = \frac{1}{T} \int_{-\infty}^{\infty} g(x) e^{-2\pi i \xi x} \, dx = \int_{-\infty}^{\infty} \mathbf{f}(y) e^{-2\pi i \xi \frac{T y}{T}} \, dy = F \mathbf{f} (\xi T)
\]

PSF-2 \Rightarrow

\[
\frac{1}{T} \sum_{k=-\infty}^{\infty} \mathbf{f} \left( \frac{k}{T} \right) e^{-2\pi i \xi \frac{k}{T}} \leq \sum_{k=-\infty}^{\infty} F \mathbf{f} (\xi + k T), \xi \in \mathbb{R}
\]

Change \( \xi \) to \( \frac{\xi}{T} \) to conclude,

\[
\frac{1}{T} \sum_{k=-\infty}^{\infty} \mathbf{f} \left( \frac{k}{T} \right) e^{-2\pi i \frac{k}{T} \xi} = \sum_{k=-\infty}^{\infty} F \mathbf{f} (\xi + k) , \xi \in \mathbb{R}
\] (2)

Replacing in (1)

\[
\frac{1}{T} \left( \sum_{k=-\infty}^{\infty} \mathbf{f} \left( \frac{k}{T} \right) e^{-2\pi i \frac{k}{T} \xi} \right) \chi_{\left[ \frac{1}{2}, \frac{1}{2} \right]} (\xi) = F \mathbf{f} (\xi)
\] (3)

Take the inverse Fourier transform of both sides

\[
\mathbf{f}(x) = \int_{-\infty}^{\infty} \frac{1}{T} \left( \sum_{k=-\infty}^{\infty} \mathbf{f} \left( \frac{k}{T} \right) e^{-2\pi i \frac{k}{T} \xi} \right) e^{2\pi i \xi _{T} x} \, dx
\]

\[
= \frac{1}{T} \sum_{k=-\infty}^{\infty} \mathbf{f} \left( \frac{k}{T} \right) \int_{-\infty}^{\infty} e^{2\pi i \left( x - \frac{k}{T} \right) \xi} \, dx
\]

\[\text{(Ex 4.1)}\]

\[
= \frac{1}{T} \sum_{k=-\infty}^{\infty} \mathbf{f} \left( \frac{k}{T} \right) \frac{\sin \pi \left( \frac{Tx}{T} \right)}{\pi \left( \frac{Tx}{T} \right)}
\]

\[\text{for } x = \frac{k}{T}\]
1.5. FAST FOURIER TRANSFORM (FFT)

(DFT) \( \hat{f}(k) = \sum_{n=0}^{N-1} f(n) e^{-\frac{2\pi i kn}{N}} \), \( d=0 \), \( k=0, \ldots, N-1 \)

DFT requires \( \approx N^2 \) operations \( \text{(additions + multiplications)} \).

**FFT** is an algorithm to compute DFT when \( N = 2^L \), and reduces the complexity of computations to \( \approx N \cdot \log_2 N \).

For even frequencies:

\[
\hat{f}(2k) = \sum_{n=0}^{N/2-1} f(n) e^{-\frac{2\pi i 2kn}{N}} + \sum_{n=N/2}^{N-1} f(n) e^{-\frac{2\pi i 2kn}{N}}
\]

\[
= \sum_{n=0}^{N/2-1} f(n) e^{-\frac{2\pi i kn}{N/2}} + \sum_{m=0}^{N/2-1} f(m+N/2) e^{-\frac{2\pi i km}{N/2}}
\]

\[
= \sum_{n=0}^{N/2-1} \left[f(n) + f(n+N/2)\right] e^{-\frac{2\pi i kn}{N/2}} \quad (6)
\]

\( \hat{f}(2k) \) can be computed using FFT of \( f_0(n) = f(n) + f(n+N/2) \in S_{N/2} \).

For odd frequencies:

\[
\hat{f}(2k+1) = \sum_{n=0}^{N/2-1} f(n) e^{-\frac{2\pi i (2k+1)n}{N}} + \sum_{n=N/2}^{N-1} f(n) e^{-\frac{2\pi i (2k+1)n}{N}}
\]

\[
= \sum_{n=0}^{N/2-1} e^{-\frac{2\pi i kn}{N/2}} f(n) e^{-\frac{2\pi i kn}{N/2}} + \sum_{m=0}^{N/2-1} e^{-\frac{2\pi i km}{N/2}} f(m+N/2) e^{-\frac{2\pi i km}{N/2}}
\]

\[
= \sum_{n=0}^{N/2-1} e^{-\frac{2\pi i kn}{N/2}} \left[f(n) - f(n+N/2)\right] e^{-\frac{2\pi i kn}{N/2}} \quad (7)
\]

\( \hat{f}(2k+1) \) can be computed using FFT for \( f_1(n) = e^{-\frac{2\pi i n}{N}} [f(n) - f(n+N/2)] \in S_{N/2} \).
1.4 DISCRETE FOURIER TRANSFORM (DFT)

"Signals are sampled and represented by numbers"

Sample rate \( T = 1 \) if \( \{ f(n) \}_{n \in \mathbb{Z}} \); in practice only \( N \) samples are taken, \( f = \{ f(n) \}_{n = 0}^{N-1} \) is a \( \text{discrete} \) \( \text{finite} \) \( \text{signal} \) of size \( N \).

Fourier transform has to be defined in this context. \( S_N \) = \( \text{discrete} \) (complex) \( \text{signals} \) of size \( N \); \( S_N \) is a Hilbert space with inner product (of \( \text{dim} \ N \))

\[
\langle f, g \rangle = \sum_{n=0}^{N-1} f(n) \overline{g(n)}
\]

(4)

Thm 4.2

\[
\{ e_k(n) \}_{k \in \mathbb{Z}} = \left\{ \left( \frac{e^{2 \pi i kn}}{N} \right) \right\}_{n=0}^{N-1}
\]

is an \( \text{ortho} \) \( \text{basis} \) of \( S_N \).

\( \therefore \) See page 1.2.3 of my notes

\[ x \]

If \( f \in S_N \), \( f = \sum_{k=0}^{N-1} \lambda_k e_k \) and \( \langle f, e_m \rangle = \sum_{k=0}^{N-1} \lambda_k \overline{e_k(m)} \)

\[
\sum_{k=0}^{N-1} \lambda_k \overline{e_k(m)} = \lambda_m \Rightarrow \lambda_m = \frac{1}{N} \sum_{k=0}^{N-1} \langle f, e_k \rangle e_k
\]

(5)

and

\[
N \rightarrow k \sum_{k=0}^{N-1} \frac{1}{N} \overline{e_k(m)} e_k
\]

\( \text{Notation: } \hat{f}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n) e^{-2\pi i kn/N} \) in DFT of \( f \)

\( k = 0, 1, \ldots, N-1 \)

(5) \( \Rightarrow \hat{f}(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{f}(k) e^{2\pi i kn/N} \) (Inverse DFT)

Ex 1.3. Plancherel: if \( f \in S_N \), \( \| f \|^2 = \frac{1}{N} \sum_{k=0}^{N-1} \left| \hat{f}(k) \right|^2 \).
\[ C(N) = \text{# of complex operations needed to calculate DFT using} \]
\[ (6) \text{ and (7) for } N = 2^e \]
\[ (6) \quad (7) \]
\[ C(N) = 2C\left(\frac{N}{2}\right) + \frac{N}{2} + \left(\frac{N}{2} + \frac{N}{2}\right) = 2C\left(\frac{N}{2}\right) + \frac{3N}{2} \]
\[ \text{and } C(1) = 0 \text{ because if } f \in S_1, \quad \hat{f}(0) = f(0) \]
\[ C(2^1) = 2C(2^{1-1}) + \frac{3}{2} \cdot 2^1 = 2 \left[ 2C(2^{2-2}) + \frac{3}{2} \cdot 2^{2-1} \right] + \frac{3}{2} \cdot 2^1 \]
\[ = 4C(2^{1-2}) + 2 \cdot \frac{3}{2} \cdot 2^1 = 4 \left[ 2C(2^{2-3}) + \frac{3}{2} \cdot 2^{2-2} \right] + 2 \cdot \frac{3}{2} \cdot 2^1 \]
\[ = 2^3 C(2^{1-3}) + 3 \cdot \frac{3}{2} \cdot 2^1 = 8 \text{ times} \]
\[ = 2^e C(1) + e \cdot \frac{3}{2} \cdot 2^1 = \frac{3}{2} \cdot e \cdot 2^1 = \frac{3}{2} \cdot N \log_2 N \]
\[ \times \]

\[ \sum_{n_1 \in N} \sum_{n_2 \in N} \sum_{n_3 \in N} \sum_{n_4 \in N} \sum_{n_5 \in N} \sum_{n_6 \in N} \sum_{n_7 \in N} \sum_{n_8 \in N} \]
LECTURE 2

2.1. INTRODUCTION TO WAVELETS

History of wavelets: From my talk at UCM3 - 2015
- \( \{ e^{2\pi i x \chi_{[k,k+1]}(x)} : k \in \mathbb{Z}, k \in \mathbb{Z} \} \) is an o.n.b. of \( L^2(\mathbb{R}) \)

\[ \frac{1}{\sqrt{n}} \sum_{j=k} \hat{f}(w) e^{2\pi i x \chi_{[k,k+1]}(x)} \] Geometric system

When \( w \in \mathbb{Z} \), \( \hat{f} \) is an o.n.b. of \( L^2(\mathbb{R}) \)?

[Balian- low '80s]: If \( \hat{f} \) is an o.n.b. of \( L^2(\mathbb{R}) \), then either
\[ \int_{-\infty}^{\infty} x^2 |\hat{f}(x)|^2 dx = \infty \] or \[ \int_{-\infty}^{\infty} \left( \sum_{j=-\infty}^{\infty} \hat{f}(j) e^{2\pi i j x} \right)^2 dx = \infty \]

Ex 2.1. Let
\[ f(x) = \begin{cases} 2x+1 & \text{if } -\frac{1}{2} \leq x < 0 \\ -2x+1 & \text{if } 0 \leq x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \]

Show that \( \hat{f} \) is not an o.n.b. of \( L^2(\mathbb{R}) \)

Translation: \( T_y f(x) = f(x - y) \); Modulation: \( M_y f(x) = e^{2\pi i xy} f(x) \)
- I.e., \( \hat{f}(y) = \{ M_{2^k}(T_y^{-1}) \hat{f}(x) : k \in \mathbb{Z} \} \)

Dilation: \( D_a f(x) = a^k f(a^{-1} x) \), \( a > 0 \).

- \( \| D_a f \|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} a^{2k} |f(a^{-1} x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(y/a)|^2 dy = \| f \|_2^2 \)

Define \( W \in L^2(\mathbb{R}) \):
\[ W(y) = \{ D_0 T_y \hat{f}(x) = f(x) : k \in \mathbb{Z} \} \]

- \( W(y) = D_0(T_{2k} f)(x) = 2^{k/2} T_{2k} f(2x) = 2^{k/2} f(2x - k) \)

Def 2.1. \( \psi \in L^2(\mathbb{R}) \) is an orthonormal wavelet if \( W(\psi) \) is an o.n.b. of \( L^2(\mathbb{R}) \).
If \( \text{supp} \, \Phi_{j,k} = [k/2^j, (k+1)/2^j] \), draw \( \Psi_{j,k} : = D_2 \Psi_{1,j,k} \) for \( j \) large in \( \mathbb{Z}^+ \) and \( j' \) large in \( \mathbb{Z}^- \).

\[
\text{Supp} \, \Psi_{j,k} : -1 \leq x - k \leq 1 \Rightarrow \frac{k-1}{2^j} \leq x \leq \frac{k+1}{2^j}.
\]

\( \Psi_{j,k} \) concentrated around \( \frac{k}{2^j} \)

\( \text{Good to detect} \)

\( \text{details of a} \)

\( \text{spectral} \)

\( \text{Very spread out} \)

\( \text{Small amplitude: Good to detect the few decay of a spectral} \)

\[x\]

2.2 Haar and Shannon Wavelets

2.2.1. The Haar Wavelet

\[ \psi(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases} \]

\( \psi \) is an orthonormal wavelet in \( L^2(\mathbb{R}) \).

- O.N. System: Several steps (See § 3.2 of my notes)
- Basis: Postpone until we know MRAS

Ex 2.2. Find the Haar coefficients \( \langle f, \psi_{j,k} \rangle \) for all \( f, \psi_{j,k} \) when \( f = X_{c_0, \xi} \).
2.2.2. THE SHANNON WAVELET

\[ \psi \text{ o t. } F\psi (t) = \chi(t) \]
\[ \mathcal{F}\chi(t) = \begin{cases} 1 & \text{for } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \]

\[ \|\psi\|_2 = 1 \]

Ex 2.3. Show that \( \psi(x) = \frac{\sin(2\pi x) - \sin(\pi x)}{\pi x} \), \( \psi(0) = 1 \)

To show, we need properties of \( \mathcal{F} \) and operators

**Prop. 2.2** \( \psi \in L^2(\mathbb{R}) \),

- a) \( \mathcal{F}(T_y \psi)(t) = M_y F\psi(t) \) \( \forall y \in \mathbb{R} \)
- b) \( \mathcal{F}(M_y \psi)(t) = T_y F\psi(t) \) \( \forall y \in \mathbb{R} \)
- c) \( \mathcal{F}(D_{a,\lambda} \psi)(t) = D_{a,\lambda} F\psi(t) \) \( \forall a > 0 \)

**Proof**

- For \( a) \)
  \[ \mathcal{F}(T_y \psi)(t) = \int_{-\infty}^{\infty} T_y \psi(x) e^{-2\pi i x t} dx = \int_{-\infty}^{\infty} \psi(x+y) e^{-2\pi i x t} dx = e^{2\pi i y t} \]

Ex 2.4. Complete the proof of Prop. 2.2.

Ex 2.5. Prove

- a) \( \langle \psi_j, \chi_{j,k} \rangle = 0 \)
- b) \( \langle \psi_j, \chi_{j,k} \rangle = 0 \) \( \forall j, k \in \mathbb{Z} \), \( \forall \chi_{j,k} \in \mathcal{F} \chi_{j,k} \)
Basis: Enough to show that for all $f \in L^2(\mathbb{R})$

\[ \|f\|_2^2 = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |<f, \psi_{j,k}>|^2 \tag{1} \]

We need:

(A) \{ e^{2\pi i 2 \gamma j} \}_{j \in \mathbb{Z}} \text{ is an orthonormal basis of } L^2(-\frac{1}{2}, \frac{1}{2}) \cup \{ \frac{1}{2}, 1 \}

(B) \sum_{j \in \mathbb{Z}} \chi_\gamma(2^{\frac{1}{3}}) = 1 \text{ for all } \gamma \in \mathbb{R} - \{ 0 \}

\[
\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |<f, \psi_{j,k}>|^2 = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |<f, \psi_{j,k}>|^2
\]

Exp. 2.2

\[
= \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \left| \int \overline{F_f(s)} e^{2\pi i \frac{2\gamma j}{s}} e^{-F_f(2^{\frac{1}{3}})ds} \right|^2
\]

\[
= \int \sum_{j \in \mathbb{Z}} \left| \int \overline{F_f(s)} e^{2\pi i \frac{2\gamma j}{s}} e^{-F_f(2^{\frac{1}{3}})ds} \right|^2 d\gamma
\]

\[
= \sum_{j \in \mathbb{Z}} \left| \int \overline{F_f(s)} e^{2\pi i \frac{2\gamma j}{s}} e^{-F_f(2^{\frac{1}{3}})ds} \right|^2 d\gamma
\]

\[
= \sum_{j \in \mathbb{Z}} \left| \int F_f(s)^2 ds \right|^2 = \sum_{j \in \mathbb{Z}} \chi_\gamma(2^{\frac{1}{3}}) |F_f(s)|^2 ds
\]

\[
= \int |F_f(s)|^2 ds = \|f\|_2^2.
\]

This proves (1).
2.3 Multiresolution Analysis and Properties

Definition 2.3: An MRA is a collection \( \{ V_j : j \in \mathbb{Z} \} \) of closed linear subspaces of \( L^2(\mathbb{R}) \) s.t.

1. \( V_j \subset V_{j+1} \) \( \forall j \in \mathbb{Z} \)
2. \( f \in V_j \Leftrightarrow \hat{f}(2^{-j} \cdot) \in V_{j+1} \)
3. \( \bigcap_{j \in \mathbb{Z}} V_j = \{ 0 \} \)
4. \( \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}) \)
5. \( \exists \psi \in V_0 \) s.t. \( \{ T_{k \cdot} \psi : k \in \mathbb{Z} \} \) o.n. basis of \( V_0 \) (Scaling)

Remark 1: (1), (2) and (5) \( \Rightarrow \) (3) [see ChWJ, Thm 1.6 in Chapter 2.1]

Remark 2: If (1), (2) and (5) hold and \( |F \psi(k)| \) is continuous at \( 0, \) then (4) \( \Rightarrow \) \( |F \psi(0)| = 1 \) [see Thm 1.7 in Chapter 2 of ChWJ]

Proposition 2.4: \( \| g \|_{L^2(\mathbb{R})} \) \( < \sum_{k \in \mathbb{Z}} \| F \psi(k \cdot) \|_2^2 = 1 \) a.e. \( g \in W_j \)

Proof: See page 3.3.3 of my Notes: Compute \( \langle g, T_k \psi \rangle_{L^2(\mathbb{R})} \).

Proposition 2.5: \( \psi \in L^2(\mathbb{R}) \) scaling function of an MRA - Let

\[ \psi_{j,k}(x) = D_{2^j \cdot k} \psi (x) = 2^{j/2} \psi (2^j x - k). \]

Then \( \{ \psi_{j,k} : k \in \mathbb{Z} \} \) is an o.n. basis of \( V_j \)

Proof: See pages 3.3.4 and 3.3.5 of my Notes.
Interpretation of Properties (3) and (4):
\[ P_{V_j} f : L^2(\mathbb{R}) \rightarrow V_j \text{ orthogonal projection. By Prop 2.5} \]
\[ P_{V_j} f = \sum_{k \in \mathbb{Z}} \langle f \rangle_{V_{j+k}} \psi_{j+k} \quad (\in L^2(\mathbb{R})) \]  
(2)

(3) \Rightarrow \lim_{j \to -\infty} ||P_{V_j} f||_2 = 0;
(4) = \lim_{j \to -\infty} ||P_{V_j} f - f||_2 = 0

\[ P_{V_j} f \] is an approximation to \( f \) when \( j \to -\infty \). (Tendency)

Let \( W_j \) be s.t. \( V_j \oplus W_j = V_{j+1} \); \( P_{V_j} + P_{W_j} = P_{V_{j+1}} \)

\[ P_{W_j} f = P_{V_{j+1}} f - P_{V_j} f \] encodes the difference of the image at different levels. This difference will be given with the wavelet to be constructed.

---

**Example 1 (Haar MRA)**

\[ V_j = \{ f \in L^2(\mathbb{R}) : f \text{ constant on } [k \frac{1}{2^j}, (k+1) \frac{1}{2^j}) \}, k \in \mathbb{Z} \}

The scaling function is \( \psi(x) = \chi_{[0,1)}(x) \)

(1) \checkmark (2) Check (5) \checkmark (3) True by Remark 1

To see (4):

\[ F \psi(s) = F(\chi_{[0,1)})(s) = e^{-i\pi s} \frac{\sin \pi s}{\pi s} \quad F \psi(0) = 1 \]

Since \( F \psi \) is continuous at \( s = 0 \), (4) follows from Remark 2.
Write formula (2) for the Haar MRA

$$\psi_{j,k} = 2^{j/2} \varphi \left( 2^j x - k \right) = 2^{j/2} \chi_{E_{2^j}, 2^{-j}} (x)$$

$$\langle f, \psi_{j,k} \rangle = \int f(x) \overline{\psi_{j,k}(x)} \, dx = \int f(x) \cdot 2^{j/2} \, dx$$

$$V_j = \sum_{k \in \mathbb{Z}} 2^j \left( \sum_{\ell \in \mathbb{Z}} f(\ell) \chi_{E_{2^j}, 2^{-j}} \right) \chi_{E_{2^j}, 2^{-j}}$$

The coefficients of this approximation are the mean value of $f$ over the interval $E_{2^j, 2^{-j}}$.

---

**Example 2 (Shannon MRA)**

Scaling function $\varphi : \mathcal{F}(f) = \chi_{E_{2^{j+1}}, 2^{-j+1}}$; $\varphi(x) = \frac{\sin \pi x}{\pi x}$

with $\varphi(0) = 1$.

$$V_0 = \text{span} \{ T_{ke} \varphi : k \in \mathbb{Z} \}.$$

Since $\sum_{k \in \mathbb{Z}} |T_{ke} \varphi|^2 = \int \varphi^2 = 1$ by Prop 2.2, \{ $\varphi(\cdot - t) : k \in \mathbb{Z}$ \} is an o.n. basis of $V_0$, which is (5).

Since $\mathcal{F}(T_{ke} \varphi) = e^{-2 \pi i ke} \mathcal{F}(\varphi)$

$$= e^{-2 \pi i ke} \chi_{E_{2^{j+1}}, 2^{-j+1}} \quad V_0 = \{ f \in L^2(\mathbb{R}) : \sup \mathcal{F} f \in L^2 \}$$

Define $V_j = \text{span} \{ \psi_{j,k} : k \in \mathbb{Z} \}$

$$\mathcal{F}(\psi_{j,k}) = \mathcal{F}(D_{2^j} T_{ke} \varphi) = D_{2^j} \mathcal{F}_{ke} \mathcal{F}(\varphi) = 2^{j/2} \mathcal{F}_{ke} \mathcal{F}(\varphi)$$

$$= 2^{j/2} e^{-2 \pi i e^2} \chi_{E_{2^j}, 2^{-j}} \quad V_j = \{ f \in L^2(\mathbb{R}) : \sup \mathcal{F} f \in L^2 \}$$

$$\therefore V_j = \{ f \in L^2(\mathbb{R}) : \sup \mathcal{F} f \in L^2 \}$$
(1) \checkmark, (2) \checkmark, (5) \checkmark  \quad (3) \quad \text{follows by Remark 1}

Since \( F_1(0) = 1 \) & \( F_1(p) \) is constant at zero, (4) follows from Remark 2.
3.1 DESIGNING WAVELETS FROM AN MRA

3.1.1 FILTERS ASSOCIATED WITH AN MRA

\((V_j)_{j \in \mathbb{Z}}, \varphi \) is an MRA for \(L^2(\mathbb{R})\). \(\frac{1}{2} \varphi(\frac{x}{2}) \in V_1 \supseteq V_0 \)

\[ \varphi(x) = \sum_{k \in \mathbb{Z}} h(k \varphi(x-k)) \text{ in } L^2(\mathbb{R}) \] \hspace{1cm} (1)

with

\[ h(k) = \langle \frac{1}{2} \varphi(\frac{x}{2}), \varphi(x-k) \rangle = \int_{\mathbb{R}} \frac{1}{2} \varphi(\frac{x}{2}) \overline{\varphi(x-k)} \, dx \] \hspace{1cm} (2)

Take \( \hat{F} \) in (1)

\[ (F \varphi)(2x) = \left( \sum_{k \in \mathbb{Z}} h(k) e^{-2\pi i kx} \right) \varphi(x) = h(x) \varphi(x) \] \hspace{1cm} (3)

and \( h(x) \) is called low pass filter of the MRA. (Also, the

transfomation function) and belongs to \( L^2(\mathbb{R}) \). since \( \{h(k)\} \in \ell^2(\mathbb{Z}) \)

Prop 3.1. The low pass filter \( h \) of an MRA satisfies

\[ |h(0)|^2 + |h(\frac{1}{2})|^2 = 1 \text{ a.e. } x \in \mathbb{R} \] If \( \| \varphi \|_2 = 1 \), then \( |h(0)| = 1 \)

p/ See page 3.4.2 of my NOTES.

Ex 3.1. Consider Haar-MRA with scaling function \( \varphi = \chi_{[0,1]} \)

Show that \( h(0) = h(\frac{1}{2}) = \frac{1}{2} \) and \( h(k) = 0 \) of \( k \neq 0, 1 \).

Deduce that

\[ h(x) = \frac{1}{2} + \frac{1}{2} e^{-2\pi i x} = e^{-i \pi x} \cos(2\pi x) \]

Ex 3.2. Consider Shannon-MRA with \( \varphi(x) = \frac{\sin(\pi x)}{\pi x}, \varphi(0) = 1 \).

Show that \( h(\frac{1}{2}) = \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x) \) in \( [-\frac{1}{2}, \frac{1}{2}] \).
3.1.2. Mallat's Recipe to Design Wavelets

Define \( W_0 = V_1 \). Define \( W_j = \{ g \in L^2(\mathbb{R}) : g \in W_0 \} \). It can be seen that \( V_j \oplus W_j = V_{j+1} \). Hence,

\[
V_{j+1} = V_j \oplus W_j = V_j \oplus W_j \oplus W_j = \cdots = \bigoplus_{k=-\infty}^{\infty} W_k \quad \text{(by (3))}
\]

Since \( \bigcup_{j} V_j = L^2(\mathbb{R}) \), \( L^2(\mathbb{R}) = \bigoplus_{k=-\infty}^{\infty} W_k \) \hspace{1cm} (4)

Strategy: Find \( \Psi \in W_0 \), i.e., \( \{ T_k \Psi : k \in \mathbb{Z} \} \) an o.n. basis of \( W_0 \). Then \( \{ T_k \Psi : k \in \mathbb{Z} \} \) is an o.n. basis of \( W_j \). By (4), \( \{ T_k \Psi : k \in \mathbb{Z} \} \) an o.n. basis of \( L^2(\mathbb{R}) \)

Properties of \( \Psi \)

(A) If \( \chi \in W_0 \Rightarrow \frac{1}{2} \chi \left( \frac{x}{2} \right) \in W_1 \subset V_0 \)

\[
\frac{1}{2} \chi \left( \frac{x}{2} \right) = \sum_{k \in \mathbb{Z}} \tilde{g}_k e^{-2\pi ikx} \quad \text{in } L^2(\mathbb{R}) \]

with \( \{ \tilde{g}_k : k \in \mathbb{Z} \} \in L^2(\mathbb{R}) \) and

\[
\tilde{g}_k = \langle \frac{1}{2} \chi \left( \frac{x}{2} \right), \psi \rangle = \int_{-\infty}^{\infty} \frac{1}{2} \chi \left( \frac{x}{2} \right) \psi (x-k) \, dx
\]

As in (3),

\[
(F \chi)(2x) = \left( \sum_{k \in \mathbb{Z}} \tilde{g}_k e^{-2\pi ik2x} \right) \chi (x) \Leftrightarrow \text{Q (5) } \quad \text{F} \chi (x) \Leftrightarrow \tilde{g}(x) \psi (x) \]

and \( \chi (x) \) is called the high-pass filter of the MRA.
(B) ** If \( \{ T_{k}: k \in \mathbb{Z} \} \) o.n. system in basis of \( W_{0} \),
\[
1g(0)1^{2} + 1g \left( \frac{1}{2} \right)1^{2} = 1 \quad \text{a.e. } 0 \in \mathbb{R} \quad (8)
\]

Same proof as the one in Prop 3.1.

(C) If \( W_{0} = \text{span} \{ T_{k}: k \in \mathbb{Z} \} \perp V_{0} = \text{span} \{ T_{k}: k \in \mathbb{Z} \} \), then
\[
\sum_{k \in \mathbb{Z}} F_{h}(x + k) g(x + k) = 0 \quad \text{a.e. } x \in \mathbb{R} \quad (9)
\]

P/ See page 3.4.9 of my notes

(D) If \( \{ T_{k}: k \in \mathbb{Z} \} \) o.n. basis of \( W_{0} \),
\[
g(0)h(0) + g \left( \frac{1}{2} \right)h \left( \frac{3}{2} \right) = 0 \quad \text{a.e. } 0 \in \mathbb{R} \quad (10)
\]

P/ See page 3.4.9 of my notes

Remark. If \( F_{h}(0) = 1 \Rightarrow h(0) = 1 \Rightarrow h \left( \frac{1}{2} \right) = 0 \Rightarrow \)
\[
g(0) = 0 \Rightarrow F_{h}(0) = 0 \Leftrightarrow \int_{0}^{\infty} h(x) dx = 0
\]

Prop 3.3 \( \{ T_{k}: k \in \mathbb{Z} \} \) o.n. basis of \( W_{0} \) \( \Leftrightarrow \) (8) \& (10)

Only \( \Rightarrow \) remains to be proved.
Thm. 3.3 (Mallat, 1989)

\((\psi, \psi^2) \text{ MRA for } L^2(\mathbb{R}) \text{ with low pass filter } h. \) Define

\[ q(\xi) = \frac{e^{-2\pi i \xi} h(\xi + \frac{1}{2})}{\sqrt{2}} \psi(\xi) \quad (11) \]

for any 1-periodic function \( \psi \) with \( |\psi(\xi)| = 1 \text{ a.e. } \xi \in \mathbb{R}. \)

Then \( \psi \) is given by

\[ \mathcal{F}_\psi(\omega) = q(\omega) \mathcal{F}_\psi(\omega) \quad \text{a.e. } \omega \in \mathbb{R}, \]

then \( \{ \psi_{k, \ell} \} = \{ \phi_{k, \ell} \} \) is an orthonormal basis of \( L^2(\mathbb{R}) \).

P. By Prop 3.2 & Estravity, enough to show (8) \& (10).

Since \( |h(\xi)|^2 \leq (h(\xi + \frac{1}{2}))^2 = 1 \text{ a.e. (Prop 3.1) } \), then

\[ q(\xi) = \frac{1}{\sqrt{2}} \mathcal{F}_\psi(\xi) = \frac{1}{\sqrt{2}} \mathcal{F}_\psi(\xi) \left[ 1 + e^{-2\pi i \xi} \right] = 0, \]

Ex 3.3. If \( q(\xi) \) is given by (11) with \( \psi(\xi) = 1 \), show that \( q_{k, \ell} = \mathcal{F}_\psi(\xi - k \ell) \) \( \psi_{k, \ell} \text{ is a basis.} \)

\[ \mathcal{F}_\psi(\omega) = q(\omega) \mathcal{F}_\psi(\omega) = \sum_{k \in \mathbb{Z}} q_{k, \ell} \psi_{k, \ell} e^{-2\pi i \omega \xi} \]
Ex 3.3
\[
\sum_{k \in \mathbb{Z}} \overline{h(1-k)} (-1)^{1-k} f(T_k \phi)(\xi) = \mathcal{F} \left( \sum_{k \in \mathbb{Z}} \overline{h(1-k)} (-1)^{1-k} T_k \phi \right)(\xi)
\]

\[
\mathcal{F} \left( \sum_{k \in \mathbb{Z}} (-1)^{1-k} h(1-k) \phi(x-k) \right)
\]

\[
\phi(x) = 2 \sum_{k=-\infty}^{\infty} (-1)^{1-k} h(1-k) \phi(2x-k)
\]

(12)

Ex 3.4. Use (12) and Ex 3.1 to show that Mallet's recipe for Haar-MRA with \( \phi(x) = \chi_{[0,1]}(x) \) gives
\[
\phi(x) = \chi_{[0,\frac{1}{2}]}(x) - \chi_{[\frac{1}{2},1]}(x)
\]

(\( \psi \))
3.5. Fast Wavelet Transform

\[ V_0 \longrightarrow V_1 \longrightarrow \ldots \longrightarrow V_{j-2} \longrightarrow V_{j-1} \longrightarrow V_j \]

Since \( \{ \phi_{j,k} : k \in \mathbb{Z} \} \) is an orthonormal basis of \( V_j \):

\[ p_{V_j} f = \sum_{k \in \mathbb{Z}} c_j(k) \phi_{j,k} \quad \text{with} \quad c_j(k) = \langle f, \phi_{j,k} \rangle \quad (13) \]

(Aproximation)

Since \( \{ \psi_{j,k} : k \in \mathbb{Z} \} \) is an orthonormal basis of \( W_j \):

\[ p_{W_j} f = \sum_{k \in \mathbb{Z}} d_j(k) \psi_{j,k} \quad \text{with} \quad d_j(k) = \langle f, \psi_{j,k} \rangle \quad (14) \]

Since \( p_{V_j} f + p_{W_j} f = p_{V_{j+1}} f \) (14) give the details.

---

Objective: knowing \( c_{j+1}(k) \) and coefficients \( h_{j+1}(k) \) and \( g_{j+1}(k) \), find formulas for \( c_{j-1}(k) \) and \( d_{j-1}(k) \) - Decomposition algorithm -

---

Lemma 3.4. \( \Psi_{j-1,p} = \sqrt{2} \sum_{k \in \mathbb{Z}} h_{j-1}(k-2p) \phi_{j-1,k} \) \( (\text{in} \ L^2(\mathbb{R})) \)

\[ \Psi_{j-1,p} = \sum_{k \in \mathbb{Z}} h_{j-1} \phi_{j-1,k} \] \( (\text{in} \ L^2(\mathbb{R})) \)

Proof: Recall

\[ \frac{1}{2} \Phi_2(\frac{x}{2}) = \sum_{k \in \mathbb{Z}} h_{2k} \Phi_2(x - 2k), \quad \text{where} \quad \psi_{j-1,p} = \frac{1}{\sqrt{2}} \Phi_2(\frac{x}{2}), \quad \Phi_2(-k) \]

Since \( \Psi_{j-1,p} \in V_j \),

\[ \Psi_{j-1,p} = \sum_{k \in \mathbb{Z}} \langle \Psi_{j-1,p}, \psi_{j,k} \rangle \psi_{j,k} \]

\[ \langle \Psi_{j-1,p}, \psi_{j,k} \rangle = \int_{-\infty}^{\infty} \frac{1}{2} \Phi_2(z-x) 2 \psi_{j-1,p}(z-k) \] \( dz \)
\[ y = 2^{j-2p} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}} \varphi \left( \frac{y}{2^j} \right) \varphi \left( y + 2^p - k \right) \, dy \]

\[ = \sqrt{2} < \frac{1}{2} \varphi \left( \frac{y}{2^j} \right), \varphi \left( y - (1 - 2p) \right) > = \sqrt{2} \sum_{k \in \mathbb{Z}} h_{[k-2p]} \mathcal{G}_j(k) \] (By *)

Now

\[ \mathcal{G}_{j-1}(p) = \mathcal{G}_j(p) \]

\[ = \sqrt{2} \sum_{k \in \mathbb{Z}} h_{[k-2p]} \mathcal{G}_j(k) = \sqrt{2} \sum_{k \in \mathbb{Z}} h_{[k-2p]} \mathcal{G}_j(k) \] (15)

which computes \( \mathcal{G}_{j-1}(p) \) in terms of \( \mathcal{G}_j(k) + h_{[k-2p]} \)

\[ \mathcal{G}_{j-1}(p) \]

Ex 3.5 Imitate the proof of (15) to show

\[ \mathcal{A}_{j-1}(p) = \sqrt{2} \sum_{k \in \mathbb{Z}} h_{[k-2p]} \mathcal{A}_j(k) \] (16)

with terms \( \mathcal{A}_j(k) \) in terms of \( \mathcal{A}_j(k) \) and \( h_{[k-2p]} \)

(15) \& (16) are called "decomposition algorithm"

Objective: knowing \( \mathcal{G}_j(k) \), \( \mathcal{A}_j(k) \) and the filters well, find a formula for \( \mathcal{G}_j(p) \) - reconstruction algorithm

Since \( \psi_j = \psi_{j-1} \oplus \psi_{j-1} \), the collection

\[ \{ \psi_{j-1,k} : k \in \mathbb{Z}, j \in \mathbb{Z} \} \]

is an o.n. basis of \( \psi_j \). Since \( \psi_j, p \in \psi_j \),
\[ Y_{j,p} = \sum_{k \in \mathbb{Z}} \langle \psi_{j,p}, \psi_{j,k} \rangle \psi_{j-1,k} + \sum_{k \in \mathbb{Z}} \langle \psi_{j,k}, \psi_{j-1,k} \rangle \psi_{j-1,k} \]

Hence,

\[ C_j(p) = \langle \phi_j, \psi_{j,p} \rangle = \]

\[ = \sqrt{2} \sum_{k \in \mathbb{Z}} h(p-2k) \psi_{j-1,k} \quad + \quad \sqrt{2} \sum_{k \in \mathbb{Z}} q(p-2k) \phi_{j-1,k} \]  \( (17) \)

(17) is the reconstruction algorithm


(15) and (16) are linear equations; they can be written in matrix form:

Let \( C_j = (c_j(k))_{k \in \mathbb{Z}} \) and \( D_j = (d_j(k))_{k \in \mathbb{Z}} \)

(15) \( \iff \) \( C_{j-1} = H C_j \); (16) \( \iff \) \( D_{j-1} = G C_{j-1} \)

where \( H \) is a double infinite matrix \( H = (h_{j-k-2p})_{p,k \in \mathbb{Z}} \)

Abusing notation

\[ (15) \iff (16) \iff \begin{pmatrix} C_{j-1} \\ D_{j-1} \end{pmatrix} = \begin{pmatrix} H & | \\ G & | \end{pmatrix} \begin{pmatrix} C_j \\ 0 \end{pmatrix} \]
(14) is also linear: can be written in matrix form

\( C_j = \begin{pmatrix} H^T & G^T \end{pmatrix} \begin{pmatrix} C_{j-1} \\ D_{j-1} \end{pmatrix} \)

\[ \text{Ex 3.6: Assume only } g_1(0), g_1(1), g_1(2), g_1(3) \text{ are non-zero and consider the Haar wavelet filters } h_{10}=\frac{1}{2}, h_{11}=\frac{1}{2} \text{ and } g_{10}=-\frac{1}{2}, g_{113}=\frac{1}{2}. \text{ Write equations (15), (16) and (17) in matrix form.} \]

\[ \begin{pmatrix} C_{j-1}(0) \\ C_{j-1}(1) \end{pmatrix} = \sqrt{2} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} C_j(0) \\ C_j(1) \\ g_{11} \\ g_{13} \end{pmatrix} \]

\[ H_{2\times 4} \]

\[ \begin{pmatrix} d_{j-1}(0) \\ d_{j-1}(1) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} C_j(0) \\ C_j(1) \\ g_{12} \\ g_{13} \end{pmatrix} \]

\[ (12) \quad C_j(0) = \sqrt{2} \frac{1}{2} C_{j-1}(0) + \sqrt{2} \frac{1}{2} d_{j-1}(0); \quad C_j(1) = \sqrt{2} \frac{1}{2} C_{j-1}(1) + \sqrt{2} \frac{1}{2} d_{j-1}(1) \]

\[ C_j(2) = \sqrt{2} \frac{1}{2} C_{j-1}(1) + \sqrt{2} \frac{1}{2} d_{j-1}(1); \quad C_j(3) = \sqrt{2} \frac{1}{2} C_{j-1}(2) + \sqrt{2} \frac{1}{2} d_{j-1}(1) \]
When sampling a signal we take $N = 2^J$ data.
Assume $(c_j(x))_{k=0}^{N-1}$ are the data of data of $C_J(x)$.
Do the decomposition algorithm $J$ times to obtain
\[ C_0(x) \quad \text{and} \quad c_j(x), \quad 0 \leq j < J, \quad 0 \leq k < 2^j \]
Descending (small) details, a compressed signal is obtained.

Assume $C_J$ has only $N = 2^J$ non-zero elements, $C_J(x), \quad 0 \leq k < N$ and $h = (h_0, h_1, \ldots, h_k)$ has only
the first $k$ well non-zero. The number of operations
to compute $C_{J-1}(p), \quad 0 \leq p < \frac{N}{2}$, in (see (15)) is $\approx 2k$.
To compute all of them, we need $\approx 2k \frac{N}{2} = kN$ operations.

To compute all $c_j(p), \quad 0 \leq p < \frac{N}{2}$, need $\approx kN$ operations.
To compute $C_{J-1}(p) \ast c_j(p)$, need $\approx 2kN$ operations.

Thus, after $J$ steps, we have done $\approx$ operations
\[ 2kN + 2k \frac{N}{2} + 2k \frac{N}{4} + \ldots + 2k \frac{N}{2^J} \leq 2kN \left( 1 + \frac{1}{2} + \frac{1}{4} + \ldots \right) \leq 4kN \]

Thus, the wavelet decomposition is an algorithm factor
than FFT (this requires $CN \log_2 N$ operations).
4.1. Properties of wavelets

Important for compressing: to have \( \| f \|_2^2 = \langle f, \psi^m \rangle \) small. The number of zero moments of a wavelet \( \psi \) with compact support makes the details small.

Prop 4.1. \( \psi \) an o.n. wavelet in \( L^2(\mathbb{R}) \) with \( \text{supp } \psi \subset [-a, a] \) and
\[
\int_{\mathbb{R}} \psi(y) dy = \int_{\mathbb{R}} x \psi(x) dx = \ldots = \int_{\mathbb{R}} x^m \psi(x) dx = 0.
\]

If \( f \in C^0, \text{exp}, \text{in an open nbh of } \frac{k \xi}{2^j}, \frac{k + \eta}{2^j} \), then
\[
|\psi(y)| = K_{\xi, \eta} y_{1/2} \leq C_{\psi, \eta} y_{1/2} d y (y^{1/2})
\]

P/ See page 3.6.1 of my notes.

Prop 4.2. If \( \psi \) is an o.n. wavelet with \( \int_{\mathbb{R}} \psi(x) dx = \int_{\mathbb{R}} x \psi(x) dx = \ldots = \int_{\mathbb{R}} x^m \psi(x) dx = 0 \), then \( g(0) = 0 \) and \( h(\frac{1}{2}) = 0 \).

P/ \[
\frac{d F_x}{d s} (\xi) = \int_{\mathbb{R}} (\xi x) \psi(x) e^{-2\pi i s x} dx \Rightarrow \frac{d F_x}{d s} (0) = 0
\]
\( \Rightarrow \int_{\mathbb{R}} x \psi(x) dx = 0 \). From \( F_x (2s) = g(s) F_x (s) \Rightarrow \)

25/08/2016
\[ \frac{d P_4}{ds} (2s) = g'(s) F_4(s) + g(s) \frac{d F_4}{ds} (s) \]

\( (8 > 0) \Rightarrow 0 = g'(s) \cdot 1 + g(s) \cdot \frac{d F_4}{ds} (s) = g'(s) \cdot 1 + 0 \)

\( \therefore g'(s) = 0 \) (We are assuming \( F_4(s) \neq 1 \)).

From \( g(s) = e^{-\frac{2\pi i s}{2}} \),

\[ g'(s) = -2\pi i e^{-\frac{2\pi i s}{2}} + e^{-\frac{2\pi i s}{2}} \cdot h'(s + \frac{1}{2}) \]

\( (8 > 0) \Rightarrow 0 = g'(s) = -2\pi i e^{-\frac{2\pi i s}{2}} + e^{-\frac{2\pi i s}{2}} \cdot h'(s + \frac{1}{2}) = 0 + h'(\frac{s}{2}) \)

\( \therefore h'(\frac{s}{2}) = 0. \)

\[ x \]

Remark: Iterating derivatives in the proof of Prop 4.2 it can be proved that if \( \int \chi^l \chi \omega dx = 0, \ l = 0, \ldots, p - 1 \), then \( h^{(p-1)}(\frac{1}{2}) = 0 \). This has the effect of making \( h(s) \) more flat at \( \frac{1}{2} \).

\[ x \]

To be able to program exactly (15), (16) and (17) (wavelet decomposition and reconstruction) need \( h_{\mathbb{R}}[k] \) and also \( g_{\mathbb{R}}[k] \) to have only a finite number of non-zero terms. This are called \textbf{finite filters}. If \( h_{\mathbb{R}}[0], \ldots, h_{\mathbb{R}}[N - 1] \neq 0 \) and \( h_{\mathbb{R}}[k] = 0 \) for \( k < 0 \) and \( k > N \), we say that \( \text{supp } h_{\mathbb{R}} \subseteq [0, N] \). In this case \( h(s) = \sum_{k=0}^{N-1} h_{\mathbb{R}}[k] e^{-2\pi i k s} \).
Compact scaling functions \( \psi \) give rise to filters of finite length and compactly supported wavelet.

**Prop 4.3.** Suppose \( \text{supp} \ \psi \subset [0, N] \). Then

1. \( h_k(t) = \sum_{-N+1 \leq k \leq 2N-1} h_k \psi_{k} e^{-2\pi i k t} \) (finite filter of length \( N+1 \))

2. \( \text{supp} \ \psi \subset \left[ -\frac{N+1}{2}, \frac{N+1}{2} \right] \) (translation compact support)

**Proof.**

We know \( h_k(t) = \frac{1}{2} \int_{-\infty}^{\infty} \psi(\frac{x}{2}) \overline{\psi(x-k)} \, dx \)

\( \text{supp} \ \psi(\frac{x}{2}) \subset [0, 2N] \) and \( \text{supp} \ \psi(x-k) \subset [k, k+N] \).

For the support to intersect: \( k+N \geq 0 \) and \( k \leq 2N \). Thus \( h_k(t) \neq 0 \) if \( -N < k < 2N \).

1. Recall \( \psi(x) = 2 \sum_{k=-\infty}^{\infty} (-1)^{k} \overline{h_k} \psi(2x-k) \)

\( h_k(t) \neq 0 \iff -N+1 \leq 1-k \leq 2N-1 \iff 2N+2 \leq k \leq N \)

\( \text{supp} \ \psi(2x-k) \subset \left[ \frac{k}{2}, \frac{k+N}{2} \right] \). 
\( \frac{k}{2} \leq x \leq \frac{k+N}{2} \)

Therefore, \( \text{supp} \ \psi \subset \left[ -\frac{2N+2}{2}, \frac{2N+2}{2} \right] = \left[ -N+1, N \right] \times \)
4.2. Properties of Filter Coefficients

Recall in matrix form the decomposition algorithm, with a little different notation. Start with a vector column \( x \) of size \( N = 2^q \), \( x = (x_0, x_1, \ldots, x_{N-1})^T \).

\[
\begin{pmatrix}
\tilde{a} \\
\tilde{d}
\end{pmatrix} = \frac{1}{\sqrt{2}}
\begin{pmatrix}
H_{N/2} \\
G_{N/2}
\end{pmatrix}
\begin{pmatrix}
x \\
\tilde{x}
\end{pmatrix}
\tag{1}
\]

Wavelet transform. Dcimp algorithm.

\( \tilde{a} \) = approximation (size=\( N/2 \)) ; \( \tilde{d} \) = details (size=\( N/2 \))

Reconstruction or inverse wavelet transform

\[
x = \sqrt{2}
\begin{pmatrix}
H_{N/2}^T \\
G_{N/2}^T
\end{pmatrix}
\begin{pmatrix}
\tilde{a} \\
\tilde{d}
\end{pmatrix}
\tag{2}
\]

The matrix \( H \) is formed with the low pass filter coefficients. \( G \) is formed with the high pass filter coefficients. If the scaling function \( \phi \) is known

\[
\phi(x) = \int_{-\infty}^{\infty} \frac{1}{2} \psi\left(\frac{x}{2}\right) \overline{\psi}\left(x-k\right) \, dx ; \quad \psi(x) = (2^{-1/2}) \phi(2^{-1}x)
\]

But it is not necessary (for finite filters) to know \( \phi \) up to be able to compute \( \tilde{d} \). It can be done using the properties of the transform. Function

\[
\phi(x) = \sum_{k \in \mathbb{Z}} h(k) \psi(2^{-1}x-k)
\]

\[
\tilde{d}(x) = \sum_{k \in \mathbb{Z}} h(k) \phi(2^{-1}x-k)
\]
Properties of \( h(f) \):

1. \( |h(f)|^2 + |h(f + \frac{1}{2})|^2 = 1 \) \hspace{1cm} (Prop 3.1)
2. \( h(0) = 1 \)
3. \( h\left(\frac{1}{2}\right) = 0 \)

These three properties are not independent. In fact, (3) follows from (1) \& (2) or (2) follows from (1) \& (3).

We shall use (1) \& (3).

\[ (3) \iff \Theta = h\left(\frac{1}{2}\right) \iff \sum_{n=0}^{\infty}(-1)^k \overline{h[k]} e^{j\pi k} \quad (4) \]

\[ (2) \iff \sum_{k=-\infty}^{\infty} h(k) = 1 \quad (5) \]

Prop 4.1 Property (1) for \( h(f) \) is equivalent to

\[ \sum_{k \in \mathbb{Z}} |h[k]|^2 = \frac{1}{2} \quad \text{(ii)} \sum_{k \in \mathbb{Z}} h[k] \overline{h[k+2]} = 0 \quad \forall n \in \mathbb{Z}^*\]

\[ \text{Proof:} \]

\[ 1 = h(f) \overline{h(f)} + h\left(\frac{1}{2}\right) \overline{h\left(\frac{1}{2}\right)} = \]

\[ = \left( \sum_{k=-\infty}^{\infty} h[k] e^{-j2\pi kf} \right) \left( \sum_{k=-\infty}^{\infty} \overline{h[k]} e^{j2\pi kf} \right) + \]

\[ + \left( \sum_{k=-\infty}^{\infty} h[k+1] e^{-j2\pi k} \right) \left( \sum_{k=-\infty}^{\infty} \overline{h[k+1]} e^{j2\pi k} \right) \]

\[ = \sum_{k=0}^{\infty} \sum_{l=-\infty}^{\infty} \left( h[k] \overline{h[l]} + (-1)^{k-l} h[k+1] \overline{h[l+1]} \right) e^{j2\pi(k-l)} \]

\[ = \sum_{k=0}^{\infty} \sum_{l=-\infty}^{\infty} \left( \sum_{m=-\infty}^{\infty} 2 \delta[h-l] \overline{h[l+2m]} \right) e^{j2\pi(l-k)} \]

Result follows by the equality of coefficients. \( \Box \)
Ex. 4.4 Suppose \( h = [h_0, h_{11}] \) — only two non-zero coefficients, and is the low pass filter of an O.N. wavelet. Write the above equations for this case and show that

\[
\begin{align*}
  h_0 &= \frac{1}{2} = h_{11} \\
  g_{10} &= -\frac{1}{2}, \\
  g_{11} &= \frac{1}{2}
\end{align*}
\]

\[
\begin{align*}
  1 &+ h_0^2 = 1/2 \\
  h_0 + h_{11} &= 1 \\
  h_0 - h_{11} &= 0
\end{align*}
\]

\[
\begin{align*}
  g_{11} &= (-1)^{1-k} h_{1-k,1} \\
  g_{10} &= -\frac{1}{2}, \\
  g_{11} &= \frac{1}{2}
\end{align*}
\]

\[
\begin{align*}
  g_{11} &= (-1)^{1-k} h_{1-k,1} = \frac{1}{2}
\end{align*}
\]
4.3 Daubechies' Wavelet of Order 4

Has 4 non-zero welfs: \(h(3) = \bar{h}(0) + \bar{h}(1)e^{-2\pi i \frac{1}{4}} + \bar{h}(2)e^{-2\pi i \frac{2}{4}} + \bar{h}(3)e^{-2\pi i \frac{3}{4}}\). The conditions of section 4.1 give:

\[
\begin{align*}
(1) \quad & h(0) + \bar{h}(1) + \bar{h}(2) + \bar{h}(3) = 1 \\
(2) \quad & h(0) + \bar{h}(1) + \bar{h}(2) = 0 \\
(3) \quad & h(0)^2 + h(1)^2 + h(2)^2 + h(3)^2 = \frac{3}{2} \\
(4) \quad & h(0) + h(2) + h(3) = 0
\end{align*}
\]

Eq. (6) follows from (7), (8) and (9) because (1) \& (3) \Rightarrow (2). We have 3 equations and 4 unknowns. Try to solve it.

\[
\begin{align*}
(9) \quad & 1 \quad h_2, h_3 \perp h_0, h_1 \Rightarrow (h_2, h_3) = c \quad (-h_3, h_0) \\
& h_2 = -c \quad h_3, \quad h_3 = c \quad h_0 \quad (10) \quad c \neq 0
\end{align*}
\]

(8) \Rightarrow

\[
\begin{align*}
& h_0^2 + h_1^2 + c^2 h_1^2 + c^2 h_0^2 = \frac{3}{2} \\
& h_0^2 + h_1^2 = \frac{4}{2(1+c^2)} \quad \text{(11)}
\end{align*}
\]

\[
\text{Convergence of radius } \frac{1}{\sqrt{2(1+c^2)}}
\]

(7) \Rightarrow

\[
\begin{align*}
& h_0 - h_1 = c \quad h_1, -c \quad h_0 = 0 \\
& h_1 = \frac{1-c}{1+c} \quad h_0 \quad c \neq -1 \quad (12) \quad \text{straight line}
\end{align*}
\]

Infinitely many solutions
Daubechies impose an additional condition

\[ h'(y) = 0 \quad \text{(filter selector at } y = \frac{1}{2}) \quad \text{(See Prop 4.2)} \]

\[ h(y) = \sum_{k \in \mathbb{Z}} h_k e^{-2\pi i k y} \Rightarrow h'(y) = -2\pi i \sum_{k \in \mathbb{Z}} k h_k e^{-2\pi i k y} \]

\[ \Rightarrow h'(\frac{1}{2}) = -2\pi i \sum_{k \in \mathbb{Z}} k h_k (-1)^k \]

(13) \[ \sum_{k \in \mathbb{Z}} k h_k (-1)^k = 0 \]

For our filter \((h_0, h_1, h_2, h_3)\)

\[-h_1 + 2h_2 - 3h_3 = 0 \quad \text{(14)}\]

From (10), \[-h_1 - 2ch_1 - 3ch_0 = 0\]

\[ h_1 = \frac{-3c}{1 + 2c} \quad \text{(15) } c = -\frac{1}{2}\]

strawhat line

To have solution, slopes of straight lines (13) and (15) must be equal

\[ \frac{1-c}{1+c} = -\frac{3c}{1+2c} \Rightarrow c = -2 \pm \sqrt{3} \]

Take \[ c = -2 + \sqrt{3} \]

(12) \[ h_1 = \frac{1-c}{1+c} h_0 = \frac{3-\sqrt{3}}{2} h_0 = \frac{(3-\sqrt{3})(1+\sqrt{3})}{-1+\sqrt{3}} h_0 \]

\[ = -\frac{2\sqrt{3}}{-2} h_0 = \sqrt{3} h_0 \quad \text{(16)} \]
\((11) \Rightarrow \quad 4 \ h_0^2 = \frac{4}{2(1+\varepsilon^2)} = \frac{1}{\varepsilon} \quad 1 + (-2\sqrt{3})^2 = \frac{2+\sqrt{3}}{8} \quad h_0^2 = \frac{2+\sqrt{3}}{8}\)

Before taking square roots, \((1+\sqrt{3})^2 = 4+2\sqrt{3} = 2(2+\sqrt{3})\).

Hence

\[ h_0^2 = \frac{(1+\sqrt{3})^2}{64} \Rightarrow h_0 = \frac{1+\sqrt{3}}{8} \]

Take, for example \( h_0 = \frac{1+\sqrt{3}}{8} \). Then,

\[(10) \quad h_1 = \frac{1-\varepsilon}{1+\varepsilon} \cdot h_0 = \frac{\sqrt{3}}{8} \quad h_2 = -\varepsilon h_1 = \frac{3-\sqrt{3}}{8} \quad h_3 = \varepsilon h_2 = \frac{1-\sqrt{3}}{8}\]

Daubechis's filter of order 4:

\[
\begin{align*}
    h_0 &= \frac{1+\sqrt{3}}{8} \quad h_1 &= \frac{3+\sqrt{3}}{8} \quad h_2 &= \frac{3-\sqrt{3}}{8} \quad h_3 &= \frac{1-\sqrt{3}}{8} \\
\end{align*}
\]

Since \( g_{\ell} = (-1)^{\ell} h_{\ell}\)

\[
\begin{align*}
    g_0 &= -\frac{3+\sqrt{3}}{8} \quad g_1 &= \frac{1+\sqrt{3}}{8} \quad g_{-1} = \frac{3+\sqrt{3}}{8} \quad g_{-2} = -\frac{1-\sqrt{3}}{8} \\
\end{align*}
\]
4.4. IMAGE PROCESSING WITH WAVELETS

A grayscale digital image often be viewed as an $M \times N$ matrix whose elements are integer numbers between 0 and $255 = 2^8 - 1$ (0 = black, 255 = white).

The number indicates the gray intensity of the pixel.

(Show example in transp. 2.4 of Tekla UC3-M)

Let $X = (X_{ij})_{i=1, j=1}^{M, N}$ be a grayscale digital image. To process $X$ with 0.n wavelets, apply wavelet transforms to the columns of $X$ to obtain $N = 2^q, M = 2^q$

\[
\sqrt{2} \begin{bmatrix} H & G \\ G & -H \end{bmatrix} X \rightarrow \begin{bmatrix} \text{approx} \\ \text{details} \end{bmatrix}
\]

Apply now the wavelet transform to the rows of the new image to obtain

\[
\sqrt{2} \begin{bmatrix} H & G \\ G & -H \end{bmatrix} X \begin{bmatrix} H^T & G^T \\ G^T & -H^T \end{bmatrix} \rightarrow \begin{bmatrix} \text{A} \\ \text{V} \\ \text{H} \\ \text{G} \end{bmatrix}
\]

\[
= 2 \begin{bmatrix} H & G \\ G & -H \end{bmatrix} \begin{bmatrix} H^T & G^T \\ G^T & -H^T \end{bmatrix} = 2 \begin{bmatrix} H_X H_X^T & H_X G_X^T \\ G_X H_X^T & G_X G_X^T \end{bmatrix}
\]

\[
= \begin{bmatrix} A & V \\ H & G \end{bmatrix}
\]

(Show transparency S.1 of Notes, UC3M-Tekla)
Ex. 4.2 Compute the approximation and the details (horizontal, vertical, and diagonal) of the image 

\[ x = (x_{ij})_{i,j=0}^3, \quad a_{i=0} = 20 \text{ if } j=0,1,2; \quad a_{i=0} = 20 \text{ if } i=0,1,2 \]

and the rest of the coefficients zero, using a 2D-Haar transform. (Use grey values with Black=20, White=0, to have an image representation.)
LECTURE 5: THE JPEG FORMAT FOR IMAGES

Brief history of JPEG and JPEG2000 = JPEG2K

Use transparencies Section 9.4 - JPEG (Old Notes)

Step 1. The RGB system for colors

| (255, 0, 0) | Red       | (255, 255, 0) | Red and Green Yellow |
| (0, 255, 0) | Green     | (0, 255, 255) | Green + Blue Cyan   |
| (0, 0, 255) | Blue      | (255, 0, 255) | Red and Blue Magenta |
| (0, 0, 0)   | Black     | (255, 255, 255) | White               |

3-D representation of RGB system

Step 2. Color space transformation

\[
\begin{align*}
Y &= 0.257 R + 0.504 G + 0.098 B \\
C_b &= -0.148 R - 0.291 G + 0.439 B \\
C_r &= 0.439 R - 0.368 G - 0.071 B
\end{align*}
\]

Other transformation may be applied

Step 3. Downsampling

Downsampling by 2 is done eliminating every second number in the components of the color space representation
4.1. 2D Discrete Cosine Transform in JPEG

Example of a grey scale representation of color of an 8x8 block of an image.

4.2. CDF-Biorthogonal wavelets (Cohen, Daubechis, Feauveau). Similar to 2D-orthogonal wavelet transform

\[ X_{8 \times 8} \xrightarrow{\text{Transform}} \hat{X}_{8 \times 8} = (\hat{X}_{ij})_{i,j=1}^8 \]

**Step 5. Quantization**

\[
\begin{bmatrix}
16 & 11 & 10 & 16 & 24 & 40 & 51 & 61 \\
12 & 12 & 14 & 19 & 26 & 58 & 66 & 74 \\
14 & 13 & 16 & 24 & 40 & 56 & 60 & 55 \\
13 & 18 & 24 & 70 & 56 & 62 & 62 & 45 \\
24 & 26 & 62 & 74 & 56 & 56 & 62 & 66 \\
49 & 92 & 101 & 84 & 74 & 66 & 72 & 65 \\
72 & 92 & 95 & 98 & 100 & 108 & 109 & 99
\end{bmatrix}
\]

\[ = (Q_{ij})_{i,j=1}^8 \]

\[ B = (b_{ij})_{i,j=1}^8, \quad b_{ij} = \left\lfloor \frac{\hat{X}_{ij}}{Q_{ij}} \right\rfloor \text{ (rounded)} \]

Show B for the play/toy example.

**Step 6. Encoding**

Arrange numbers in a zig-zag order.

Use Huffman encoding to compress; assign a code with fewer bits to symbols that appear more frequently. No information is lost in this step.
5.1. **Discrete Cosine Transform**

$S_N^r$ = space of discrete signals of size $N$ : $(f(n))_{n=0}^{N-1}$

In Thm 1.2 (lecture 1) we show:

$$\{ e^{i2\pi kn/N} \}_{k=0}^{N-1} = \left\{ \left( \frac{1}{N^n} \right)^{\frac{1}{i2\pi kn/N}} \right\}_{n=0}^{N-1} \quad (1)$$

is an o.n. basis of $S_N^r \cong \mathbb{C}^N$

(Section 2.2.4 of my notes)

$(f(n))_{n=1}^N$ is extended by symmetry with respect to $-\frac{1}{2}$ to obtain a signal of size $2N$ given by

$$\tilde{f}(n) = \begin{cases} f(n) & \text{if } 0 \leq n \leq N-1 \\ f(n-N) & \text{if } -N \leq n \leq -1 \end{cases}$$

In $S_{2N}$, the collection

$$\{ e^{i2\pi kn/2N} \}_{k=-N}^{N-1} = \left\{ \frac{1}{\sqrt{2N}} \left( e^{i\frac{2\pi kn}{2N}} \right) \right\}_{k=-N}^{N-1} \quad (2)$$

is an o.n. basis.

Ex 5.1: Show that $S_{2N}$ (2) is an o.n. basis of $S_{2N} \cong \mathbb{C}^{2N}$ using the sum of an arithmetic prog.
Thus, any \( g \in \mathbb{F}_{2N} \), can be written as

\[
g(n) = \sum_{k=-N}^{N-1} a_k \alpha_k(n), \quad n = -N, \ldots, N-1
\]

Use \( e^{i\theta} = \cos \theta + i \sin \theta \) to show that each \( g \in \mathbb{F}_{2N} \) can be written as a linear combination of

\[
\left\{ \left( C_{2N}^{(2N)} \right)^{n-1}_{k=-N} = \left( \cos \frac{kn\pi}{N}(n+\frac{1}{2}) \right)^{n-1}_{k=-N} \right\}_{k=-N}^{N-1}
\]

\[
\cup \left\{ \left( S_{2N}^{(2N)} \right)^{n-1}_{k=-N} = \left( \sin \frac{kn\pi}{N}(n+\frac{1}{2}) \right)^{n-1}_{k=-N} \right\}_{k=-N}^{N-1}
\]

Collection (3) has \( 4N \) elements, while \( \mathbb{F}_{2N} \) has \( 2N \) elements. Thus, half of them have to be linear combinations of the other half:

\[
S_0^{(2N)} = (0) ; \quad C_0^{(2N)} = (1) ; \quad S_{-N}^{(2N)} = (-(-1)^n)_{n=-N}^{N-1} ; \quad C_N^{(2N)} = (0)
\]

If \( 0 < k < N \)

\[
\left( C_{-k}^{(2N)} \right)^{n-1}_{n=-N} = \left( C_k^{(2N)} \right)^{n-1}_{n=-N}
\]

\[
\left( S_{-k}^{(2N)} \right)^{n-1}_{n=-N} = - \left( S_k^{(2N)} \right)^{n-1}_{n=-N}
\]

Thus, each element of \( \mathbb{F}_{2N} \) can be written as a linear combination of

\[
\left\{ \left( C_{2N}^{(2N)} \right)^{n-1}_{k=-N} = \left( \cos \frac{kn\pi}{N}(n+\frac{1}{2}) \right)^{n-1}_{k=-N} \right\}_{k=-N}^{N-1}
\]

\[
\cup \left\{ \left( S_{2N}^{(2N)} \right)^{n-1}_{k=-N} = \left( \sin \frac{kn\pi}{N}(n+\frac{1}{2}) \right)^{n-1}_{k=-N} \right\}_{k=-1}^{N-1}
\]
In terms of Eqn. (4), \( \tilde{f} \in S_{2N} \) as

\[
\tilde{f}(n) = \sum_{k=0}^{N-1} a_k \xi_k(n) + \sum_{k=1}^{N} b_k S_k(n), \quad -N \leq n \leq N-1
\]

The discrete signals \( S_k(n) \), \( n = -N, \ldots, N \), are antisymmetric w.r.t. \(-\frac{1}{2}\):

\[
S_k(-1-n) = \sin \frac{k\pi}{2N} (-1-n+\frac{1}{2}) = \sin \frac{k\pi}{2N} (-n + \frac{1}{2}) = -S_k(n)
\]

Thus, since \( (\tilde{f}(n))_{n=-N}^{N-1} \) is symmetric w.r.t. \(-\frac{1}{2}\), we conclude \( b_k = 0, \quad k = 1, \ldots, N \).

Since \( (f(n))_{n=0}^{N-1} = (\tilde{f}(n))_{n=0}^{N-1} \), any \( f \in S_N \) can be written as a linear combination of \( \{ C_k^{(2N)}(n) \}_{n=0}^{N-1} \), \( k = 0, 1, \ldots, N-1 \).

**Thm 5.1 (DC -I basis)**

The collection

\[
\{ \sqrt{\frac{2}{N}} (C_k^{(2N)}(n))_{n=0}^{N-1} \} = \{ \sqrt{\frac{2}{N}} (\cos \frac{k\pi}{N}(n+\frac{1}{2}))_{n=0}^{N-1} \}_{k=0}^{N-1}
\]

with \( f_k = \left\{ \begin{array}{ll} 1 & \text{if } k = 0 \\ \frac{1}{2} & \text{if } 1 \leq k \leq N-1 \end{array} \right. \) is an o.n. basis of \( S_N \).

**Ex. 5.2** Show that

\[
\sum_{n=0}^{N-1} \omega_n (\frac{n}{N}(n+\frac{1}{2})) = 0, \quad k = 1, 2, \ldots, 2N-1 \quad (5)
\]

(Use the sum of a geom. progression, § 2.24 of Notes.)
\( (5) \Rightarrow c^{(2m)}_0 \perp c^{(2m)}_k, \quad k = 0, 2, 4, \ldots, N - 1. \)

To show \( c^{(2m)}_0 \perp c^{(2m)}_k, \quad 1 \leq k \leq N - 1 \):

\[
\langle c^{(2m)}_k, c^{(2m)}_e \rangle = \sum_{n=0}^{N-1} \omega_n \frac{\sin \left(\frac{\pi}{N} n + \frac{\pi}{2} \right)}{N} \omega_n \frac{\sin \left(\frac{\pi}{N} n + \frac{\pi}{2} \right)}{N} = \frac{1}{2} \sum_{n=0}^{N-1} \omega_n \left( \frac{\sin \left(\frac{\pi}{N} n + \frac{\pi}{2} \right)}{N} + \omega_n \frac{\sin \left(\frac{\pi}{N} n + \frac{\pi}{2} \right)}{N} \right) = 0
\]

Now

\[
\| \sqrt{\frac{2}{N}} c^{(2m)}_0 \| \cdots \| = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} 1 = 1
\]

\[
\| \sqrt{\frac{2}{N}} c^{(2m)}_k \| = \frac{1}{N} \sum_{n=0}^{N-1} \omega_n \frac{\sin \left(\frac{\pi}{N} n + \frac{\pi}{2} \right)}{N} = 1
\]

Using DCT-I basis (Thm 5.1), for any \( f \in F_N \):

\[
f(n) = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} \hat{f}(k) \frac{\sin \left(\frac{\pi}{N} n + \frac{\pi}{2} \right)}{N}, \quad n = 0, 1, \ldots, N - 1
\]

where

\[
\hat{f}(k) = \langle f, \sqrt{\frac{2}{N}} c^{(2m)}_k \rangle = \frac{\sin \left(\frac{\pi}{N} n + \frac{\pi}{2} \right)}{N} \sum_{n=0}^{N-1} f(n) \omega_n \frac{\sin \left(\frac{\pi}{N} n + \frac{\pi}{2} \right)}{N}
\]

This is called DCT-I transform, and (5) to inverse DCT-I transform.
(6) and (7) can be written in matrix form to use in MATLAB

\[
\begin{pmatrix}
\hat{f}_0(0) \\
\hat{f}_1(1) \\
\vdots \\
\hat{f}_{N-1}(N-1)
\end{pmatrix} =
\begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\
\omega & \omega & \cdots & \omega \\
\vdots & \vdots & \ddots & \vdots \\
\omega^{N-1} & \omega^{N-2} & \cdots & \omega^{N-1}
\end{bmatrix}
\begin{pmatrix}
f(0) \\
f(1) \\
\vdots \\
f(N-1)
\end{pmatrix}
\]

\[C_I \] 

(8) \leftrightarrow 

\[
\begin{pmatrix}
f(0) \\
\vdots \\
f(N-1)
\end{pmatrix} = C_I^T \begin{pmatrix}
\hat{f}_0(0) \\
\vdots \\
\hat{f}_{N-1}(N-1)
\end{pmatrix}.
\]

Thus,

\[
(f(n))_{n=0}^{N-1} = C_I^T (\hat{f}_x(k)) = C_I^T C_I (f(n))_{n=0}^{N-1} \Rightarrow
\]

\[C_I^T C_I = I \] i.e., \( C_I \) is an orthogonal matrix

Remark. The number of operations required to compute DCT-I well be \( \approx 2N^2 \). As in the case of FT, there is a fast algorithm to allow to compute \( \hat{f}_I(k) \), \( k=0, \ldots, N-1 \), with \( \approx N \log_2 N \) operations.
5.2 2D - DISCRETE COSINE TRANSFORM: FOR IMAGES

For each 8 x 8 block of an image, the basis used is

\[
(\frac{3}{8} \omega \frac{k}{8} (n + \frac{1}{2}) \omega \frac{l}{8} (m + \frac{1}{2}))^T, \quad \text{with } 0 \leq k, l \leq N - 1
\]

and

\[
2^p = \begin{cases} \frac{3}{8} & \text{if } p = 0 \\ 0 & \text{if } 1 \leq p \leq N - 1 \end{cases}
\]

In matrix form, the DCT coefficients are computed with

\[
(\hat{f}_{ij}(k, l))^T = C_{ij} \left( f(n, m) \right)^T C_{ij}^T
\]

The first element of the matrix is

\[
\hat{f}_{ij}(0, 0) = \frac{1}{2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f(n, m) \quad \text{(mean value)}
\]

and is called the DC coefficient of the image. It is a large coefficient. In JPEG, 128 is subtracted from the values of each pixel \{0, ..., 255\} to obtain numbers between \{-128, ..., 127\}.
5.3 Biorthogonal Filters for JPEG2000

Daubechies proved that the only symmetric, finite length orthogonal filter is the Haar filter \( h = \left[ \frac{1}{2}, \frac{1}{2} \right] \).

To find symmetric filters, Cohen–Daubechies–Feauveau relinquish orthogonality. They look for two finite low pass filters \( h(\xi) \) and \( \tilde{h}(\xi) \)

\[
\tilde{h}(\xi) \overline{h(\xi)} + \tilde{h}(\xi + \frac{1}{2}) \overline{h(\xi + \frac{1}{2})} = 1 \quad (8)
\]

\[
\tilde{h}(0) = 1, \quad h(0) = 1 \quad (9)
\]

\[
h(\frac{1}{2}) = 0, \quad \tilde{h}(\frac{1}{2}) = 0 \quad (10)
\]

Then define

\[
\tilde{g}(\xi) = e^{2\pi i \xi} \frac{\overline{h(\xi + \frac{1}{2})}}{h(\xi)} \quad (11)
\]

\[
g(\xi) = e^{2\pi i \xi} \frac{\overline{\tilde{h}(\xi + \frac{1}{2})}}{\tilde{h}(\xi)} \quad (12)
\]

are the high pass filters, to have

\[
\tilde{g}(\xi) \overline{\tilde{g}(\xi + \frac{1}{2})} + \tilde{g}(\xi + \frac{1}{2}) \overline{g(\xi + \frac{1}{2})} = 1
\]

\[
\tilde{g}(0) = 0, \quad g(0) = 0
\]

\[
\tilde{g}(\frac{1}{2}) = 1, \quad g(\frac{1}{2}) = 1
\]
Ex. 5.3  As in Ex. 3.3, deduce from (11), (12) that
\[
\hat{q}_k = (-1)^k \hat{h}_{-1-k} \quad \text{and} \quad \hat{q}_k = (-1)^k \hat{h}_{-1-k}
\]
where
\[
\hat{h}(\xi) = \sum_{k=-\infty}^{\infty} \hat{h}_k e^{2\pi i k \xi} \quad \hat{q}(\xi) = \sum_{k=-\infty}^{\infty} \hat{q}_k e^{2\pi i k \xi}
\]

Clearly,
\[
\hat{h}(\frac{1}{2}) = 0 \iff \sum_{k=-\infty}^{\infty} (-1)^k \hat{h}_k = 0 \quad (13)
\]
\[
\hat{h}(\frac{1}{2}) = 0 \iff \sum_{k=-\infty}^{\infty} (-1)^k \hat{h}_k = 0 \quad (14)
\]

and
\[
(8) \iff \sum_{k=-\infty}^{\infty} \hat{h}_k \hat{h}_{k-2n} = \frac{1}{2} \delta_{0,n} \quad n \in \mathbb{Z} \quad (15)
\]

The idea of CDF also to use a simple symmetric filter for \( \hat{h}(\xi) \); for example,
\[
\hat{h}(\xi) = \frac{1}{4} e^{2\pi i \xi} + \frac{1}{4} e^{2\pi i 2\xi} \quad \hat{h}_2 = [1, 2, 1]
\]

(\( = \frac{1}{2}(1 + \cos 2\pi \xi) \)) and find a filter \( h \) using (14) and (15).
For example, look for
\[ h = (h_2, h_1, h_0, h_1, h_2) \]
of length 5 and symmetric. (14) and (15) give
\[
\begin{align*}
\begin{cases}
  h_0 - 2h_1 + 2h_2 = 0 \\
  h_0 + h_1 &= 1 \\
  h_1 + 2h_2 &= 0
\end{cases}
\end{align*}
\]
(n=2)

Solution: \[ h_0 = \frac{3}{4}, \ h_1 = \frac{1}{4}, \ h_2 = -\frac{1}{8} \]

This is called a \textit{CDF(5, 3)} filter.

JPEG 2000 uses \textit{CDF(9, 7)} filter. The filter
\[ \hat{h} \]
of length 7
\[ \hat{h} = \frac{1}{2^5} (1, 6, 15, 20, 15, 6, 1) \]

(normalize) binomial coefficients \( \binom{6}{k} \frac{1}{2^6}, \ k = 0, \ldots, 6 \)
and find \[ h = (h_4, h_3, h_2, h_1, h_0, h_1, h_2, h_3, h_4) \]
symmetric of length 9 using (14), (15).
Processing digital images \( X = (x_{i,j})^{M \times N} \), \( M = 2^q \), \( N = 2^l \) with biorthogonal wavelets is similar to the case of orthonormal wavelets. (See § 4.4.)

First apply the wavelet transform \( W_{1/2} = \sqrt{2} \left[ \begin{array}{c} H_{1/2} \\ \dot{G}_{1/2} \end{array} \right] \)
to the columns of \( X \) to obtain

\[ \sqrt{2} \left[ \begin{array}{c} H_{1/2} \\ \dot{G}_{1/2} \end{array} \right] \times \]

Then apply the wavelet transform \( \tilde{W}_{1/2} = \sqrt{2} \left[ \begin{array}{c} \tilde{H}_{1/2} \\ \tilde{G}_{1/2} \end{array} \right] \)

for the matrix to obtain

\[ \sqrt{2} \left[ \begin{array}{c} H_{1/2} \\ \dot{G}_{1/2} \end{array} \right] \times \left[ \begin{array}{c} \tilde{H}_{1/2}^T \\ \tilde{G}_{1/2}^T \end{array} \right] \sqrt{2} = \left[ \begin{array}{cc} A & \mathcal{I} \\ \mathcal{H} & \mathcal{G} \end{array} \right] \]

Compression is done by quantizing the coefficient (after a few iterations) and use encoding in an efficient way, such as Huffman encoding.

\[ \left[ \begin{array}{c} \mathcal{M} \\ \mathcal{G} \end{array} \right] \times \]
5.4. ENCODING (HUFFMAN)

Suppose that after applying DCT and Biorthogonal CDF(7,9) to an 8x8 block of an image we have obtained:

```
26  3  6  2  2  70  0  0
 0  2  4  1  1  0  0  0
 3  1  5  1  1  0  0  0
 4  1  2  1  0  0  0  0
 0  0  0  0  0  0  0  0
 0  0  0  0  0  0  0  0
 0  0  0  0  0  0  0  0
 0  0  0  0  0  0  0  0
```

Each number is modified with 1 byte = 8 bits

- $26 = 00011010_2$; $3 = 00000011_2$;
- $6 = 00000110_2$; $2 = 00000010_2$;
- $4 = 00000100_2$; $5 = 00000101_2$;
- $0 = 00000000_2$; $1 = 00000001_2$

The bit-stream save in the computer is:

0001101000000011000000000000001100000010 000001100000010 ........

It do not necessary to separate bytes (words) since each one has 8 bits.
Hoffman found a way to modify a collection of "words" so that the bit per pixel (bpp) count is greatly reduced.

**Step 1** Find the frequency of each binary digit

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>00011010</td>
<td>1</td>
<td>$\frac{1}{64} = 0.015625 = p_0$</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>2</td>
<td>$\frac{2}{64} = 0.03125 = p_1$</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>1</td>
<td>$\frac{4}{64} = 0.0625 = p_2$</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
<td>$\frac{8}{64} = 0.125 = p_3$</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>2</td>
<td>$\frac{16}{64} = 0.25 = p_4$</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>1</td>
<td>$\frac{32}{64} = 0.5 = p_5$</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>8</td>
<td>$\frac{64}{64} = 1 = p_6$</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>45</td>
<td>$\frac{45}{64} = 0.703125 = p_7$</td>
</tr>
</tbody>
</table>

**Step 2** Order the relative frequencies in increasing order:

\[
( p_0 \leq p_2 \leq p_5 \leq p_1 \leq p_4 \leq p_3 \leq p_6 \leq p_7 )
\]
Step 3 Coding

\[ \begin{align*}
26 &= x_0 = \underline{11110} \\
3 &= x_1 = \underline{1000} \\
6 &= x_2 = \underline{11110} \\
2 &= x_3 = \underline{1110} \\
4 &= x_4 = \underline{1101} \\
5 &= x_5 = \underline{11110} \\
1 &= x_6 = 10 \\
0 &= x_7 = 0
\end{align*} \]

Coded bit stream

\[ \begin{align*}
\underline{111110} & \underline{1100} \underline{1100} \underline{1110} \underline{1110} \underline{1100} \\
\underline{1110} & \underline{1100} \underline{1100} \underline{1110} \underline{1110} \underline{1100}
\end{align*} \]

BPP per pixel

<table>
<thead>
<tr>
<th>Digit</th>
<th>Frequency</th>
<th>Bits</th>
<th>Total Bits</th>
</tr>
</thead>
<tbody>
<tr>
<td>26</td>
<td>1</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>2</td>
<td>16</td>
</tr>
<tr>
<td>0</td>
<td>45</td>
<td>1</td>
<td>45</td>
</tr>
</tbody>
</table>

\[ \text{bpp} = \frac{110}{64} = 1.719 \quad \text{(Much better than 8 bpp)} \]
Ex. 5.4 Consider the 4x4 image whose intensity matrix

\[
\begin{array}{cccc}
100 & 100 & 120 & 100 \\
100 & 50 & 50 & 40 \\
100 & 40 & 40 & 50 \\
120 & 120 & 100 & 100
\end{array}
\]

(a) Generate the Huffman code tree for this image.
(b) Write the bit stream for the image using the Huffman code (Zig-Zag)
(c) Compute the bpp of this bit stream.

Ex. 5.5 (Van Fleet, pg 95)

Given the Huffman code \( g = 10, \ 0 = 01, \ 0 = 00, \)
space key = 110, \( e = \text{1110}, \ i = \text{1111}, \)
draw the Huffman code tree and decode:

1000111011011010001110110110100001110

5/16

\[
\begin{array}{c}
0 = \text{00} \\
g = 10 \\
1 = \text{11} \\
e = \text{1110} \\
i = \text{1111}
\end{array}
\]

101111011101101100111011101110100001110

Going, going, gone
Ex 5.6. Given the Huffman code: D = 010, E = 10, H = 110, N = 011, T = 111, ___ = space, key = 00, draw the Huffman code tree and decode the bit stream sequence: 111110100010011010

THE END
THE END

T = 111
H = 110
E = 10
N = 011
D = 010
U = 00

111110100010011010

THE END
Ex 5.4.

100 - 7 \(-\frac{3}{6}\)
120 - 3 \(-\frac{3}{6}\)
50 - 3 \(-\frac{3}{6}\)
40 - 3 \(-\frac{3}{6}\)
\[
\frac{31}{16} \approx 2.6 \text{pp.}
\]

00001101001101111010111111

110000