Democracy functions of wavelet bases in general Lorentz spaces

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Abstract

We compute the democracy functions associated with wavelet bases in general Lorentz spaces \( \Lambda^q_w \) and \( \Lambda^{q,\infty}_w \), for general weights \( w \) and \( 0 < q < \infty \).

1 Introduction

The Lorentz space \( \Lambda^q_w(\mathbb{R}^d) \) is defined as the set of all measurable \( f : \mathbb{R}^d \to \mathbb{C} \) such that

\[
\|f\|_{\Lambda^q_w} := \left[ \int_0^\infty |f^*(t)|^q w(t) \, dt \right]^{1/q} < \infty,
\]

where \( f^* \) is the decreasing rearrangement of \( f \) (with respect to the Lebesgue measure) and \( w \) is a positive locally integrable function with the property \( \int_0^\infty w(s) \, ds = \infty \). We shall assume \( q \in (0, \infty) \).

Special examples include the classical \( L^{p,q}(\mathbb{R}^d) \) spaces (corresponding to \( w(t) = t^{q/p-1} \)), and the so called Lorentz-Zygmund spaces \( L^{p,q}(\log L)^r \), \( r \in \mathbb{R} \), for which \( w(t) = t^{q/p-1}(1 + |\log t|)^r \) (see [1]). More general weights \( w \) give rise to larger families such as the Lorentz-Karamata spaces, and various other examples considered in the literature (see eg [7]).

In this note we shall be interested in the efficiency of the greedy algorithm [9] for the \( N \)-term wavelet approximation of functions in \( \Lambda^q_w \). It is known that greedy algorithms with wavelet bases are never optimal in rearrangement invariant spaces, except for the \( L^p \) classes; see [10]. However, it is possible to quantify the efficiency of the algorithm in a space \( X \) by computing the so called lower and upper democracy functions; that is,

\[
h_\ell(N) = \inf_{\# \Gamma = N} \left\| \sum_{Q \in \Gamma} \psi_Q \right\|_X \quad \text{and} \quad h_r(N) = \sup_{\# \Gamma = N} \left\| \sum_{Q \in \Gamma} \psi_Q \right\|_X,
\]

where \( \{\psi_Q\} \) is a wavelet system indexed by the set \( D \) of all dyadic cubes of \( \mathbb{R}^d \). Indeed, a precise expression for \( h_\ell \) and \( h_r \) gives rise to optimal inclusions for the approximation classes \( A^q(\mathbb{X}) \) in terms of discrete Lorentz spaces (see [4]).

It is not always an easy matter to compute explicitly the democracy functions \( h_\ell \) and \( h_r \) in non-democratic settings. We refer to [3] for the case of Orlicz spaces \( L^\Phi \), and more.

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recently to [5] for the Lorentz classes \( L^{p,q} \). The objective of this note is to present the computation of \( h_\ell \) and \( h_r \) for the larger family of general Lorentz spaces \( \Lambda^q_w \).

As usual, using wavelet theory one can transfer the problem to the discrete setting. We define the space \( \lambda^q_w \) consisting of all sequences \( s = \{ s_Q \}_{Q \in D} \) such that

\[
\| s \|_{\lambda^q_w} := \left\| \left( \sum_{Q \in D} |s_Q|^2 \right)^{1/2} \right\|_{\Lambda^q_w} < \infty .
\]  

(1.3)

It is known that sufficiently regular wavelet bases in \( \mathbb{R}^d \) give an isomorphism between \( \Lambda^q_w \) and \( \lambda^q_w \) (when the Boyd indices of \( \Lambda^q_w \) are strictly between 0 and 1; see [8]). Thus studying the democracy of wavelet bases in \( \Lambda^q_w \) is equivalent to determine

\[
h_\ell(N) = \inf_{\# \Gamma = N} \left\| \sum_{Q \in \Gamma} e_Q \right\|_{\lambda^q_w} \quad \text{and} \quad h_r(N) = \sup_{\# \Gamma = N} \left\| \sum_{Q \in \Gamma} e_Q \right\|_{\lambda^q_w},
\]

where \( \{ e_Q \} \) denotes the canonical basis in \( \lambda^q_w \). We shall assume in the rest of the paper that \( h_\ell \) and \( h_r \) always refer to these quantities (which are comparable to the ones in (1.2) for \( X = \Lambda^q_w \), at least when the wavelet characterization holds).

To state our results we need some notation. We denote the primitive of \( w \) by

\[
W(t) := \int_0^t w(s) \, ds, \quad t \geq 0.
\]

Recall that \( \Lambda^q_w \) is quasi-normed if and only if \( W \) is doubling (see e.g. [2, 2.2.13]), so we shall always assume to be in this situation. Observe also that for all measurable \( E \subset \mathbb{R}^d \) we have

\[
\| \chi_E \|_{\Lambda^q_w} = W(|E|)^{1/q}.
\]

That is, \( W(t)^{1/q} \) is the fundamental function of the rearrangement invariant function space \( \Lambda^q_w \). We shall denote by \( H^+_W(t) \) the \( \text{dilation functions} \) associated with \( W \), that is

\[
H^+_W(t) := \sup_{s>0} \frac{W(ts)}{W(s)} \quad \text{and} \quad H^-_W(t) := \inf_{s>0} \frac{W(ts)}{W(s)}.
\]

Since \( W \) is doubling these are finite functions. Observe also that \( H^-(t) = 1/H^+(1/t) \).

Finally we denote by \( i_W \) the lower dilation index of \( W \) (see [6] or (2.14) for a precise definition), which we typically assume to be positive. Our results can be stated as follows.

**Theorem 1.4** Assume \( i_W > 0 \). Then for all \( N \in \mathbb{N} \) we have

\[
h_\ell(N) \approx \inf \left\{ \left( \sum_{j \in \mathbb{Z}} \frac{W(n_j 2^jd)}{W(2^jd)} \right)^{1/q} : n_j \in \mathbb{N} \cup \{0\} \text{ with } \sum_j n_j = N \right\} \quad (1.5)
\]

and

\[
h_r(N) \approx \sup \left\{ \left( \sum_{j \in \mathbb{Z}} \frac{W(n_j 2^jd)}{W(2^jd)} \right)^{1/q} : n_j \in \mathbb{N} \cup \{0\} \text{ with } \sum_j n_j = N \right\}, \quad (1.6)
\]

where the constants involved in “\( \approx \)” are independent of \( N \).
Our second result gives a more explicit expression for weights which are monotonic near 0 and $\infty$, that is, in intervals $(0, a)$ and $(b, \infty)$, for some $a \leq b$. Observe that most examples arising in practice do actually satisfy this property.

**Theorem 1.7** Assume that $w$ is monotonic near 0 and $\infty$, and that $i_W > 0$. Then for all $N \in \mathbb{N}$

$$h_\ell(N) \approx \min \{ N, H_W^-(N) \}^{1/q} \quad \text{and} \quad h_r(N) \approx \max \{ N, H_W^+(N) \}^{1/q}. \quad (1.8)$$

In particular

- (a) $w$ increasing implies $h_\ell(N) \approx N^{1/q}$ and $h_r(N) \approx H_W^+(N)^{1/q}$;
- (b) $w$ decreasing implies $h_\ell(N) \approx H_W^-(N)^{1/q}$ and $h_r(N) \approx N^{1/q}$.

Finally, we consider the weak versions of the Lorentz spaces $\Lambda_{\mathbb{R}^d}^q$. We write $\Lambda_{\mathbb{R}^d}^q, \infty$ for the set of all $f$ such that

$$\|f\|_{\Lambda_{\mathbb{R}^d}^q, \infty} := \sup_{t > 0} t^q w(f^* > t)^{1/q} = \sup_{s > 0} f^*(s) W(s)^{1/q} < \infty, \quad (1.9)$$

where $0 < q < \infty$. The corresponding sequence space $\lambda_{\mathbb{R}^d}^q, \infty$ is defined as in (1.3) with $\Lambda_{\mathbb{R}^d}^q, \infty$ in place of $\Lambda_{\mathbb{R}^d}^q$. Then we have the following

**Theorem 1.10** Assume $i_W > 0$. Then for all $N \in \mathbb{N}$ we have

$$h_\ell(N; \lambda_{\mathbb{R}^d}^q, \infty) \approx 1 \quad \text{and} \quad h_r(N; \lambda_{\mathbb{R}^d}^q, \infty) \approx H_W^+(N)^{1/q}.$$

Section 2 contains some preliminaries about $\Lambda_{\mathbb{R}^d}^q$ spaces. The proofs of the theorems are presented, respectively, in sections 3, 4 and 5. Finally, section 6 contains some examples.

## 2 Preliminaries

We need a few elementary properties about the spaces $\Lambda_{\mathbb{R}^d}^q$. First of all, it is well known that the (quasi) norm in $\Lambda_{\mathbb{R}^d}^q$ can also be written as

$$\|f\|_{\Lambda_{\mathbb{R}^d}^q} = \left[ \int_0^\infty q t^{q-1} W(\lambda_f(t)) \, dt \right]^{1/q} \quad (2.1)$$

where $\lambda_f(t) = \text{meas} \{ x \in \mathbb{R}^d : |f(x)| \geq t \}$ (see eg [2, Prop 2.2.5]). From here it is clear that

$$f \leq g \implies \|f\|_{\Lambda_{\mathbb{R}^d}^q} \leq \|g\|_{\Lambda_{\mathbb{R}^d}^q}. \quad (2.2)$$

We also need discretized versions of (2.1). Let $\mathcal{A}$ denote the collection of all sequences ${a_j}_{j=-\infty}^{\infty}$ of positive real numbers such that

$$\inf \frac{a_{j+1}}{a_j} > 1 \quad \text{and} \quad \sup \frac{a_{j+1}}{a_j} < \infty. \quad (2.3)$$

Clearly $\{a^j\}_{j \in \mathbb{Z}}$ with $a > 1$ satisfies these requirements, but we shall make use of more general examples later on. Observe that, in particular, the left condition in (2.3) implies

$$\lim_{j \to -\infty} a_j = 0 \quad \text{and} \quad \lim_{j \to +\infty} a_j = +\infty. \quad (2.4)$$
LEMMA 2.5 Let \( \{a_j\} \in A \). Then
\[
\|f\|_{A^q_w} \approx \left[ \sum_{j \in \mathbb{Z}} a_j^q W(\lambda_f(a_j)) \right]^{1/q}.
\] (2.6)

PROOF: Call \( m = \inf \frac{a_{j+1}}{a_j} \) and \( M = \sup \frac{a_{j+1}}{a_j} \). Then, from (2.1) we obtain
\[
\|f\|_{A^q_w}^q = \sum_{j \in \mathbb{Z}} \int_{a_j}^{a_{j+1}} q t^{q-1} W(\lambda_f(t)) \, dt \leq \sum_{j \in \mathbb{Z}} \int_{a_j}^{a_{j+1}} q t^{q-1} \, dt W(\lambda_f(a_j))
\]
\[
= \sum_{j \in \mathbb{Z}} (a_{j+1}^q - a_j^q) W(\lambda_f(a_j)) \leq (M^q - 1) \sum_{j \in \mathbb{Z}} a_j^q W(\lambda_f(a_j)).
\]

For the converse inequality one argues similarly
\[
\|f\|_{A^q_w}^q \geq \sum_{j \in \mathbb{Z}} (a_{j+1}^q - a_j^q) W(\lambda_f(a_{j+1})) \geq (1 - m^{-q}) \sum_{j \in \mathbb{Z}} a_j^q W(\lambda_f(a_{j+1})).
\]

\[\square\]

In the next lemma we need to use the doubling property \( W(2t) \leq c W(t) \). Since \( W \) is increasing, this property is equivalent to the subadditivity of \( W \) (with the same constant \( c \))
\[
W(s + t) \leq c(W(s) + W(t)), \quad \forall s, t > 0.
\]

Denote by \( D_W \) the smallest such constant, that is
\[
D_W = \sup_{s,t > 0} \frac{W(s + t)}{W(s) + W(t)}.
\] (2.7)

Also, for a fixed \( m > 1 \), we shall denote by \( A_m \) the subset of all sequences in \( A \) with
\[
\inf_{j \in \mathbb{Z}} \frac{a_{j+1}}{a_j} \geq m.
\] (2.8)

LEMMA 2.9 Let \( \{a_j\} \in A_m \) with \( m > D_W^{1/q} \). If \( f \in A_w^q \), then
\[
\|f\|_{A^q_w} \approx \left[ \sum_{j \in \mathbb{Z}} a_j^q W(\lambda_f(a_j : a_{j+1})) \right]^{1/q},
\] (2.10)

where \( \lambda_f(a_j : a_{j+1}) = \text{meas} \{ x \in \mathbb{R}^d : a_j \leq |f(x)| < a_{j+1} \} \).

PROOF: Using \( \lambda_f(a_j) = \lambda_f(a_j : a_{j+1}) + \lambda_f(a_{j+1}) \) and the subadditivity of \( W \) we obtain
\[
W(\lambda_f(a_j)) \leq D_W \left[ W(\lambda_f(a_j : a_{j+1})) + W(\lambda_f(a_{j+1})) \right].
\] (2.11)

Call
\[
I = \left( \sum_{j \in \mathbb{Z}} a_j^q W(\lambda_f(a_j)) \right)^{1/q} \quad \text{and} \quad II = \left( \sum_{j \in \mathbb{Z}} a_j^q W(\lambda_f(a_j : a_{j+1})) \right)^{1/q}.
\]
Clearly \( II \leq I \). For the converse, using (2.11) and \( \inf a_{j+1}/a_j \geq m \), we see that
\[
I^q \leq D_W II^q + D_W \sum_{j \in \mathbb{Z}} a_j^q W(\lambda f(a_{j+1})) \\
\leq D_W II^q + D_W \sum_{j \in \mathbb{Z}} \frac{a_{j+1}^q}{m^q} W(\lambda f(a_{j+1})) = D_W II^q + \frac{D_W}{m^q} I^q.
\]
Since we are assuming \( m^q > D_W \) it follows that
\[
\left(1 - \frac{D_W}{m^q}\right) I^q \leq D_W II^q.
\]
Thus \( I \approx II \) and the result follows from Lemma 2.5.

A similar argument gives

**Lemma 2.12** Let \( \{a_j\} \in \mathcal{A}_m \) with \( m > D_W^{1/q} \). If \( f \in \Lambda^{q,\infty}_W \) then
\[
\|f\|_{\Lambda^{q,\infty}_W} \approx \sup_{j \in \mathbb{Z}} a_j W(\lambda f(a_j : a_{j+1}))^{1/q}.
\] (2.13)

Recall from [6, p. 53] that the lower dilation index of \( W \) is defined by
\[
i_W := \sup_{0 < t < 1} \frac{\log H_W^+(t)}{\log t} = \lim_{t \to 0} \frac{\log H_W^+(t)}{\log t} = \lim_{u \to \infty} \frac{\log H_W^-(u)}{\log u}.
\] (2.14)

In the paper we will assume that \( i_W > 0 \), which implies that for all \( \epsilon > 0 \)
\[
W(su) \geq C_{\epsilon} u^{i_W - \epsilon} W(s), \quad \forall s > 0, \quad \forall u \geq 1,
\] (2.15)
for some \( C_{\epsilon} > 0 \). In §3 we shall be interested in applying Lemma 2.9 to the sequence \( a_j = W(2^{-jd})^{-1/q} \). This sequence clearly satisfies (2.4) (since we assume \( \int_0^\infty w(s)ds = \infty \)), but the validity of (2.3) depends on the growth of \( W \). We show below how to handle this under the assumption \( i_W > 0 \).

**Proposition 2.16** Assume that \( i_W > 0 \). Then the norm equivalences in (2.6), (2.10) and (2.13) hold for the sequence
\[
a_j = \frac{1}{W(2^{-jd})^{1/q}}, \quad j \in \mathbb{Z}.
\]

The proposition will be an easy consequence of the following lemma.

**Lemma 2.17** Assume that \( i_W > 0 \) and fix \( m > D_W^{1/q} \). Then there exists \( L_0 \in \mathbb{N} \) such that for every subsequence \( \{k_j\}_{j \in \mathbb{Z}} \) with the property
\[
k_{j+1} = k_j + L_0, \quad \forall j \in \mathbb{Z},
\]
the sequence \( \{W(2^{-kd})^{-1/q}\}_{j \in \mathbb{Z}} \) belongs to \( \mathcal{A}_m \).
PROOF: Call \( b_j = W(2^{-k_jd})^{-1/q} \). By the monotonicity of \( W \) and (2.15) we see that
\[
\left( \frac{b_j+1}{b_j} \right)^q \leq \frac{W(2^{-k_jd})}{W(2^{-d(k_{j+1}+L_0)})} \geq C \left( 2^{dL_0} \right)^{iW^{-\epsilon}}.
\]
It suffices to choose \( \epsilon = iW/2 \) and \( L_0 \) large enough so that the right hand side is \( \geq m^q \). The bound from above follows from the doubling property of \( W \).

**PROOF of Proposition 2.16:** We shall only prove (2.10), since the other cases are similar. Let \( L_0 \) be as in the previous lemma. Then, for each \( r \in \{0, \ldots, L_0 - 1\} \), the sequence \( a^{(r)}_j = \{ a_{jL_0+r} = W(2^{-(jL_0+r)d})^{-1/q} \}_{j \in \mathbb{Z}} \) belongs to \( A_m \). Thus, for each such \( r \) Lemma 2.9 implies that
\[
\|f\|_{A^q_m} \approx \left[ \sum_{j \in \mathbb{Z}} a^q_{jL_0+r} W(\lambda_f(a_{jL_0+r} : a_{(j+1)L_0+r})) \right]^{1/q}, \tag{2.18}
\]
for every \( f \in A^q_m \). We first show the inequality “\( \lesssim \)” for which we choose \( r = 0 \) in (2.18). By the subadditivity of \( W \), there is a constant \( C = C(W, L_0) \) such that
\[
W(\lambda_f(a_{jL_0} : a_{(j+1)L_0})) \leq C \sum_{s=0}^{L_0-1} W(\lambda_f(a_{jL_0+s} : a_{jL_0+s+1})).
\]
Inserting this into (2.18) (with \( r = 0 \)) and using \( a_{jL_0} \approx a_{jL_0+s} \) (by the doubling property of \( W \)) we easily obtain
\[
\|f\|_{A^q_m}^q \lesssim \sum_{s=0}^{L_0-1} \sum_{j \in \mathbb{Z}} a^q_{jL_0+s} W(\lambda_f(a_{jL_0+s} : a_{jL_0+s+1})) = \sum_{k \in \mathbb{Z}} a^q_k W(\lambda_f(a_{k} : a_{k+1})).
\]
Conversely, since \( L_0 \) is a finite constant \( (2.18) \) implies that
\[
\|f\|_{A^q_m}^q \approx \sum_{r=0}^{L_0-1} \sum_{j \in \mathbb{Z}} a^q_{jL_0+r} W(\lambda_f(a_{jL_0+r} : a_{(j+1)L_0+r})) \geq \sum_{r=0}^{L_0-1} \sum_{j \in \mathbb{Z}} a^q_{jL_0+r} W(\lambda_f(a_{jL_0+r} : a_{jL_0+r+1})) = \sum_{k \in \mathbb{Z}} a^q_k W(\lambda_f(a_{k} : a_{k+1})). \tag*{\Box}
\]
Finally we state a key “linearization” lemma which holds when \( iW > 0 \).

**Lemma 2.19** Suppose \( iW > 0 \). For every finite collection \( \Gamma \subset \mathcal{D} \), and every \( x \in \bigcup_{Q \in \Gamma} Q \) it holds
\[
\left( \sum_{Q \in \Gamma} \frac{\chi_Q(x)}{W((Q))^{3/2}} \right)^{1/2} \approx \frac{\chi_{Q_x}(x)}{W((Q_x))^{1/3}}, \tag{2.20}
\]
where \( Q_x \) denotes the smallest cube in \( \Gamma \) containing \( x \).

Such linearization arguments have been used by various authors in the context of \( N \)-term wavelet approximation. For an elementary proof and references see e.g. [3, §4.2.1].
3 Proof of Theorem 1.4

Let $\Gamma \subset \mathcal{D}$ with $\#\Gamma = N$. We use the notation

$$1_\Gamma = \sum_{Q \in \Gamma} \frac{e_Q}{\|e_Q\|_{\Lambda^q}} \quad \text{and} \quad S_\Gamma(x) = \left( \sum_{Q \in \Gamma} \frac{\chi_Q(x)}{W(|Q|)^{\frac{q}{2}}} \right)^{1/2}.$$

Observe from (1.3) that

$$\|e_Q\|_{\Lambda^q} = |Q|^{-1/2} \|\chi_Q\|_{\Lambda^q} = |Q|^{-1/2} W(|Q|)^{1/q},$$

so we are led to estimate the expression

$$\|1_\Gamma\|_{\Lambda^q} = \left\| \left( \sum_{Q \in \Gamma} \frac{\chi_Q(x)}{W(|Q|)^{\frac{q}{2}}} \right)^{1/2} \right\|_{\Lambda^q} = \|S_\Gamma\|_{\Lambda^q}.$$

Using (2.6) we see that

$$\|1_\Gamma\|_{\Lambda^q} \approx \left[ \sum_{j \in \mathbb{Z}} a_j^q W(|\{\|S_\Gamma\| \geq a_j\}|) \right]^{1/q}.$$

We choose $a_j = W(2^{-jd})^{-1/q}$ and denote $\Gamma_j = \{Q \in \Gamma : |Q| = 2^{-jd}\}, j \in \mathbb{Z}$. Clearly $S_\Gamma(x) \geq a_j$ for all $x \in \bigcup_{Q \in \Gamma_j} Q$, which implies

$$\|1_\Gamma\|_{\Lambda^q} \gtrsim \left[ \sum_{j \in \mathbb{Z}} \frac{W(\bigcup_{Q \in \Gamma_j} Q)}{W(2^{-jd})} \right]^{1/q} = \left[ \sum_{j \in \mathbb{Z}} \frac{W(2^{-jd}\#\Gamma_j)}{W(2^{-jd})} \right]^{1/q}.$$

For the estimate from above we use Lemma 2.19 and denote by $F_\Gamma(x)$ the function on right hand side of (2.20). Then (2.10) gives

$$\|1_\Gamma\|_{\Lambda^q} \approx \|F_\Gamma\|_{\Lambda^q} \approx \left[ \sum_{j \in \mathbb{Z}} a_j^q W(|\{a_j \leq |F_\Gamma| < a_{j+1}\}|) \right]^{1/q},$$

where as before we set $a_j = W(2^{-jd})^{-1/q}$. Then the condition $a_j \leq F_\Gamma(x) < a_{j+1}$ implies that $x \in \bigcup_{Q \in \Gamma_j} Q$, and therefore

$$\|1_\Gamma\|_{\Lambda^q} \lesssim \left[ \sum_{j \in \mathbb{Z}} \frac{W(2^{-jd}\#\Gamma_j)}{W(2^{-jd})} \right]^{1/q}.$$

We conclude that

$$\|1_\Gamma\|_{\Lambda^q} \approx \left[ \sum_{j \in \mathbb{Z}} \frac{W(2^{-jd}\#\Gamma_j)}{W(2^{-jd})} \right]^{1/q}, \quad (3.1)$$

and since $\sum \#\Gamma_j = \#\Gamma = N$, this clearly implies (1.6).
4 Proof of Theorem 1.10

The proof for the spaces $\Lambda_{\mu}^{q,\infty}$ is similar. First observe from the norm definitions that
\[ \|e_Q\|_{\Lambda_{\mu}^{q,\infty}} = |Q|^{-1/2} \|\chi_Q\|_{\Lambda_{\mu}^{q,\infty}} = |Q|^{-1/2} W(|Q|)^{1/q}, \]
so we are led to estimate the expression
\[ \|1_{\Gamma}\|_{\Lambda_{\mu}^{q,\infty}} = \left\| \left( \sum_{Q \in \Gamma} \chi_Q(x) \right) \right\|_{\Lambda_{\mu}^{q,\infty}} = \|S_{\Gamma}\|_{\Lambda_{\mu}^{q,\infty}}. \]

The lower bound $h_{\ell}(N) \geq 1$ is trivial. To see the optimality, choose $\Gamma$ formed by pairwise disjoint cubes all of different sizes. Using (2.13) with $a_j = W(2^{-jd})^{-1/q}$ we easily see that
\[ \|1_{\Gamma}\|_{\Lambda_{\mu}^{q,\infty}} \approx \sup_{j \in \mathbb{Z}} a_j W(|\{a_j \leq S_{\Gamma}(x) < a_{j+1}\}|)^{1/q} = 1, \]
which proves the assertion.

To obtain bounds for $h_r(N)$, we use again (2.13) with $a_j = W(2^{-jd})^{-1/q}$, together with Lemma 2.19, so that
\[ \|1_{\Gamma}\|_{\Lambda_{\mu}^{q,\infty}} \approx \sup_{j \in \mathbb{Z}} a_j W(|\{a_j \leq S_{\Gamma}(x) < a_{j+1}\}|)^{1/q} \leq \sup_{j \in \mathbb{Z}} \left[ \frac{W(2^{-jd}\#\Gamma_j)^{1/2}}{W(2^{-jd})} \right]^{1/q} \leq \sup_{j \in \mathbb{Z}} H_{W}^+(\#\Gamma_j)^{1/q} \leq H_{W}^+(N)^{1/q}. \]
This proves that $h_r(N) \lesssim H_{W}^+(N)^{1/q}$. For the converse, choose $\Gamma$ consisting of $N$ pairwise disjoint cubes all of the same size, say $s_0$. Then,
\[ \|1_{\Gamma}\|_{\Lambda_{\mu}^{q,\infty}} = \left\| \frac{1}{W(s_0)^{1/q}} \chi_{\cup Q \in \Gamma} \right\|_{\Lambda_{\mu}^{q,\infty}} = \frac{W(Ns_0)^{1/q}}{W(s_0)^{1/q}}. \]
We can select $s_0$ so that the last quantity is comparable to $H_{W}^+(N)^{1/q}$, concluding the proof.

5 Proof of Theorem 1.7

We say that $W$ is of type (A) if for some $c \geq 0$ and $C > 0$ it holds
\[
\begin{align*}
W(t_0) &\leq C W(t_1), & \text{for } 0 < t_0 < t_1 \leq 2c \quad (A_1) \\
W(t_1) &\leq C W(t_0), & \text{for } c/2 < t_0 < t_1 < \infty. \quad (A_2)
\end{align*}
\]
We say that $W$ is of type (B) if for some $c \geq 0$ and $C > 0$
\[
\begin{align*}
W(t_0) &\leq C W(t_1), & \text{for } 0 < t_0 < t_1 \leq 2c \quad (B_1) \\
W(t_1) &\leq C W(t_0), & \text{for } c/2 < t_0 < t_1 < \infty. \quad (B_2)
\end{align*}
\]
These conditions can easily be phrased in terms of convexity of $W$. Namely, when $c > 0$ type (A) is the same as $W$ being (quasi) convex for small $t$ and (quasi) concave for large $t$. Similarly for type (B), with opposite convexities in $W$. Observe that the exact value of the constant $c > 0$ is irrelevant, since we are assuming that $W$ is doubling. By allowing the case $c = 0$ we consider also the situations when $W$ is everywhere quasi-concave (type A), or everywhere quasi-convex (type B) in the half line $(0, \infty)$.

**Lemma 5.1** If $w$ is monotonic near 0 and $\infty$, then $W$ is either of type (A) or of type (B) for some $c \geq 0$.

**Proof:** The proof is standard, using the inequalities
\[
\min \left\{ \frac{x}{u}, \frac{y}{v} \right\} \leq \frac{x + y}{u + v} \leq \max \left\{ \frac{x}{u}, \frac{y}{v} \right\}, \quad x, y, u, v > 0.
\]
Indeed, assume that $w$ is increasing in $(0, a)$. Then for $0 < t_0 < t_1 < a$
\[
\frac{W(t_1)}{t_1} = \frac{\int_{t_0}^{t_1} w(s) ds + \int_{t_0}^{t_1} w(s) ds}{t_0 + (t_1 - t_0)} \geq \min \left\{ \frac{1}{t_0} \int_{t_0}^{t_1} w(s) ds, \frac{1}{t_1 - t_0} \int_{t_1}^{t_0} w(s) ds \right\} = \frac{W(t_0)}{t_0},
\]
where in the last step we use that, by the monotonicity of $w$,
\[
\frac{1}{t_1 - t_0} \int_{t_0}^{t_1} w(s) ds \geq w(t_0) \geq \frac{1}{t_0} \int_{t_0}^{t_1} w(s) ds.
\]
Similarly, if we assume $w$ decreasing in $(b, \infty)$ then for $t_1 > t_0$
\[
\frac{W(t_1)}{t_1} \leq \max \left\{ \frac{1}{t_0} \int_{t_0}^{t_1} w(s) ds, \frac{1}{t_1 - t_0} \int_{t_1}^{t_0} w(s) ds \right\},
\]
so if we take $t_0 > 2b$ the monotonicity of $w$ gives
\[
\frac{1}{t_1 - t_0} \int_{t_0}^{t_1} w(s) ds \leq w(t_0) \leq \frac{1}{t_0 - b} \int_{t_0}^{t_1} w(s) ds \leq 2 \frac{1}{t_0} \int_{t_0}^{t_1} w(s) ds = 2 \frac{W(t_0)}{t_0}.
\]
Using the doubling property of $W$, these inequalities can be extended respectively to the larger intervals $(0, 4b)$ and $(a/4, \infty)$, perhaps with multiplicative constants, from which it follows that $W$ is of type (A). The other cases are proved similarly.

The main result in this section is the following.

**Proposition 5.2** Assume that $W$ is of type (A) or (B) for some $c \geq 0$. Then for all $N$ and $n_j \in \mathbb{N} \cup \{0\}$ such that $\sum_{j \in \mathbb{Z}} n_j = N$ we have
\[
\min \left\{ N, H^c_W(N) \right\} \preceq \sum_{j \in \mathbb{Z}} \frac{W(n_j 2^j d)}{W(2^j d)} \preceq \max \left\{ N, H^c_W(N) \right\}, \quad (5.3)
\]
with the involved constants independent on $N$ and $n_j$.

Observe that the upper and lower bounds in (5.3) are best possible. Indeed, taking all $n_j \in \{0, 1\}$ the middle expression is exactly equal to $N$. On the other hand, taking $n_{j_0} = N$ and $n_j = 0$ for $j \neq j_0$, an appropriate choice of $j_0$ makes the middle expression comparable to $H^c_W(N)$. Thus, Theorem 1.7 is a consequence of Theorem 1.4 and Proposition 5.2 (see also Remarks 5.6 and 5.7 below).
5.1 Proof of Proposition 5.2

Assume first that $W$ is of type (A) for some $c > 0$. For simplicity, throughout the proof we shall write $\lambda_j = 2^d$. Define the sets of indices

$$J_+ = \{ j \in \mathbb{Z} : n_j \lambda_j \geq c/2 \} \quad \text{and} \quad J_- = \{ j \in \mathbb{Z} : n_j \lambda_j < c/2 \}. \quad (5.4)$$

Then using (A2) in the first inequality

$$C \sum_{j \in J_+} \frac{W(n_j \lambda_j)}{W(\lambda_j)} \geq \sum_{j \in J_+} n_j \frac{W(N \lambda_j)}{NW(\lambda_j)} \geq H^-(N) \sum_{j \in J_+} n_j/N.$$ 

Similarly, using (A1) one obtains

$$C \sum_{j \in J_-} \frac{W(n_j \lambda_j)}{W(\lambda_j)} \geq \sum_{j \in J_-} n_j.$$ 

Since either \( \sum_{j \in J_+} n_j \geq N/2 \) or \( \sum_{j \in J_-} n_j \geq N/2 \), it follows that

$$\sum_{j \in \mathbb{Z}} \frac{W(n_j \lambda_j)}{W(\lambda_j)} \geq \frac{1}{2C} \min\{N, H^-(N)\}.$$ 

To prove the upper bounds we need three sets of indices

$$J_a = \{ j : \lambda_j \geq c \}, \quad J_b = \{ j : \lambda_j < c/N \}, \quad J_c = \{ j : c/N \leq \lambda_j < c \}. \quad (5.5)$$

As before, using respectively (A2) and (A1) we see that

$$\sum_{j \in J_a} \frac{W(n_j \lambda_j)}{W(\lambda_j)} \leq C \sum_{j \in J_a} n_j$$

and

$$\sum_{j \in J_b} \frac{W(n_j \lambda_j)}{W(\lambda_j)} \leq C \sum_{j \in J_b} n_j \frac{W(N \lambda_j)}{NW(\lambda_j)} \leq C H^+(N) \sum_{j \in J_b} n_j/N.$$ 

For indices $j \in J_c$ we use the cruder estimate

$$\sup_{t > 0} \frac{W(t)}{t} \leq CW(c)/c,$$

which together with (A1) in the second step leads to

$$\sum_{j \in J_c} \frac{W(n_j \lambda_j)}{W(\lambda_j)} \leq C \sum_{j \in J_c} \frac{n_j \lambda_j W(c)}{cW(\lambda_j)} \leq C^2 \sum_{j \in J_c} \frac{n_j W(c)}{NW(c/N)} \leq C^2 H^+(N) \sum_{j \in J_c} n_j/N.$$ 

Combining the three cases we see that

$$\sum_{j \in \mathbb{Z}} \frac{W(n_j \lambda_j)}{W(\lambda_j)} \leq C^2 \left( N + H^+(N) \right) \lesssim \max\{N, H^+(N)\}.$$
REMARK 5.6 The proof just given is also valid for $W$ of type (A) with $c = 0$. In fact, in this case the sets $J_+$, $J_b$, and $J_c$ are empty, so one actually obtains

$$H^-(N) \lesssim \sum_{j \in \mathbb{Z}} \frac{W(n_j \lambda_j)}{W(\lambda_j)} \lesssim N.$$

This corresponds to the case $w$ decreasing, as stated in (b) of Theorem 1.7.

We now turn to the case when $W$ is of type (B), assuming for simplicity $c > 0$. Using the same sets $J_\pm$ as in (5.4) together with (B2) and (B1), respectively, we obtain

$$\sum_{j \in J_+} \frac{W(n_j \lambda_j)}{W(\lambda_j)} \leq C \sum_{j \in J_+} \frac{n_j W(N \lambda_j)}{N W(\lambda_j)} \leq C H^+(N) \sum_{j \in J_+} n_j/N \quad \text{and}$$

$$\sum_{j \in J_-} \frac{W(n_j \lambda_j)}{W(\lambda_j)} \leq C \sum_{j \in J_-} n_j.$$

Summing up we get

$$\sum_{j \in \mathbb{Z}} \frac{W(n_j \lambda_j)}{W(\lambda_j)} \leq 2C \max\{N, H^+(N)\}.$$

We turn to the lower bound, for which we use the sets $J_a$, $J_b$, and $J_c$ in (5.5). As before, the first two sets are easily handled with (B2) and (B1)

$$C \sum_{j \in J_a} \frac{W(n_j \lambda_j)}{W(\lambda_j)} \geq \sum_{j \in J_a} n_j \quad \text{and}$$

$$C \sum_{j \in J_b} \frac{W(n_j \lambda_j)}{W(\lambda_j)} \geq \sum_{j \in J_b} n_j \frac{W(N \lambda_j)}{N W(\lambda_j)} \geq H^-(N) \sum_{j \in J_b} n_j/N.$$

For indices $j \in J_c$ we use

$$C \inf_{t > 0} W(t)/t \geq W(c)/c,$$

which together with (B1) in the second step leads to

$$C \sum_{j \in J_c} \frac{W(n_j \lambda_j)}{W(\lambda_j)} \geq \sum_{j \in J_c} \frac{n_j \lambda_j W(c)}{c W(\lambda_j)} \geq \frac{1}{c} \sum_{j \in J_c} \frac{n_j W(c)}{N W(e/N)} \geq \frac{1}{c} H^-(N) \sum_{j \in J_c} n_j/N.$$

Now, since either $\sum_{j \in J_a} n_j \geq N/2$ or $\sum_{j \in J_b \cup J_c} n_j \geq N/2$, it follows that

$$\sum_{j \in \mathbb{Z}} \frac{W(n_j \lambda_j)}{W(\lambda_j)} \geq \frac{1}{2C^2} \min\{N, H^-(N)\}.$$

REMARK 5.7 As before, the proof is also valid for $c = 0$, obtaining in this case

$$N \lesssim \sum_{j \in \mathbb{Z}} \frac{W(n_j \lambda_j)}{W(\lambda_j)} \lesssim H^+(N).$$

This corresponds to the situation of $w$ increasing, as stated in (a) of Theorem 1.7.
6 Examples

We illustrate some examples of Lorentz weights to which the results of Theorem 1.7 can be applied. Consider the following general class of weights

\[ w(t) = \begin{cases} 
  t^{a_0 - 1} \left[ \log(e + \frac{1}{t}) \right]^{\beta}, & 0 < t \leq 1 \\
  t^{a_1 - 1} \left[ \log(e + t) \right]^{\gamma}, & t \geq 1 
\end{cases} \]

where \( a_0, a_1 > 0 \) and \( \beta, \gamma \in \mathbb{R} \). These are typical examples of piecewise monotonic weights with different behavior near 0 and \( \infty \). Observe that

\[ W(t) \approx \begin{cases} 
  t^{a_0} \left[ \log(e + \frac{1}{t}) \right]^{\beta}, & 0 < t \leq 1 \\
  t^{a_1} \left[ \log(e + t) \right]^{\gamma}, & t \geq 1. 
\end{cases} \]

From this expression it is not difficult to compute \( H^\pm_W(N) \). Indeed, a straightforward (but slightly tedious) calculation gives

\begin{enumerate}
  \item if \( a_0 < a_1 \) then \( H^-(N) \approx N^{a_0} / [\log(e + N)]^{\beta+} \) and \( H^+(N) \approx N^{a_1} [\log(e + N)]^{\gamma+} \)
  \item if \( a_0 = a_1 \) then \( H^-(N) \approx N^{a_0} / [\log(e + N)]^{\beta+ + \gamma-} \) and \( H^+(N) \approx N^{a_0} [\log(e + N)]^{\beta- + \gamma+} \)
  \item if \( a_0 > a_1 \) then \( H^-(N) \approx N^{a_1} / [\log(e + N)]^{\gamma-} \) and \( H^+(N) \approx N^{a_0} [\log(e + N)]^{\beta-} \)
\end{enumerate}

where for a real number \( x \) we denote

\[ x^+ = \begin{cases} 
  |x|, & \text{if } x \geq 0 \\
  0, & \text{if } x < 0
\end{cases} \quad \text{and} \quad x^- = \begin{cases} 
  0, & \text{if } x \geq 0 \\
  |x|, & \text{if } x < 0
\end{cases} \]

See eg [3, §3] for similar examples. In particular, setting \( a_0 = a_1 = q/p \) and \( \beta = \gamma = rq \) we obtain for the Lorentz-Zygmund spaces \( L^{p,q}(\log L)^r \)

\[ h_{t}^e(N) \approx \min \left\{ N^{\frac{1}{q}}, N^{\frac{1}{p}} [\log(e + N)]^{-|\frac{r}{p}|} \right\} \quad \text{and} \quad h_{r}^e(N) \approx \max \left\{ N^{\frac{1}{q}}, N^{\frac{1}{p}} [\log(e + N)]^{|\frac{r}{p}|} \right\}. \]

When \( r = 0 \) we recover the results for the classical \( L^{p,q} \) spaces from [5].

A second class of weights to which Theorem 1.7 is applicable is

\[ w(t) = t^{\alpha - 1} \exp(|\ln t|^\delta), \quad \alpha > 0 \quad \text{and} \quad \delta \in (0,1). \]

Observe that the functions \( \exp(|\ln t|^\delta) \) grow faster than \( |\ln t|^N \) for all \( N \) but are smaller than any power \( t^\varepsilon \) (for \( t \) near \( \infty \)) or \( 1/t^\varepsilon \) (for \( t \) near \( 0 \)). It is not difficult to see that*

\[ W(t) \approx t^\alpha \exp(|\ln t|^\delta). \quad (6.1) \]

From here one easily computes

\[ H^+_W(t) \approx t^\alpha e^{\ln t^\delta} \quad \text{and} \quad H^-_W(t) \approx t^\alpha e^{-\ln t^\delta}, \quad t > 0. \]

In particular, if \( \alpha = q/p \) we obtain for the corresponding space \( \Lambda^q_w \)

\[ h_{t}^e(N) \approx \min \left\{ N^{\frac{1}{q}}, N^{\frac{1}{p}} e^{-\frac{|\ln N|^\delta}{\delta}} \right\} \quad \text{and} \quad h_{r}^e(N) \approx \max \left\{ N^{\frac{1}{q}}, N^{\frac{1}{p}} e^{-\frac{|\ln N|^\delta}{\delta}} \right\}. \]

Observe that these spaces \( \Lambda^q_w \) are contained in all the Lorentz-Zygmund spaces \( L^{p,q}(\log L)^r \) for all \( r > 0 \) (hence also in \( L^{q,q} \)).

*In fact, if \( i_W > 0 \) it is always true that \( W(t) \approx \int_0^t W(s) s^{-1} ds \); see eg [6, p. 57].
References


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