Ph. D. thesis: Modular hyperelliptic curves

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Abstract

The purpose of this Ph. D. thesis is to generalize the so called *Shimura-Taniyama-Weil Conjecture* to nonelliptic curves. The above conjecture stablished that for every elliptic curve defined over $\mathbb{Q}$ there is a cover defined over $\mathbb{Q}$ from some modular curve $X_0(N)$. Recently, this conjecture has been completely proved by Christophe Breuil, Brian Conrad, Fred Diamond and Richard Taylor following Andrew Wiles' works.

Once this conjecture has been proved, it seems natural to determine other families of curves defined over $\mathbb{Q}$ that are modular, where we say that a curve is modular if it admits a nonconstant morphism defined over $\mathbb{Q}$ from some modular curve $X_1(N)$ onto the curve. This is our starting point. To be exact, our goal is the study of the hyperelliptic curves that are modular.

The study of curves defined over $\mathbb{Q}$ of genus greater than one (not necessarily hyperelliptic) that are modular is very different from the study of elliptic curve defined over $\mathbb{Q}$, since the elliptic curves are canonically isomorphic to their jacobians and as abelian varieties they are $\mathbb{Q}$-simple. So, for every modular genus one curve defined over $\mathbb{Q}$ there is a cover defined over $\mathbb{Q}$ from some modular curve $X_1(N)$ such that the corresponding morphism between their jacobians factorices through the new part of the jacobian of $X_1(N)$. However, this condition is not always true for curves of genus greater that one. We say that a curve is *new modular of level* N if satisfies the above condition. This family of curves contains all the elliptic curves defined over $\mathbb{Q}$, and our study will be the new modular curves that are hyperelliptic.

In this thesis we have proved that the set of new modular hyperelliptic curves is finite. This surprising result led us to the determination of these curves, that is, to find equations and the corresponding modular morphisms.

For this goal we have bounded their possible genus and we have found conditions over their corresponding levels. We are able to determine that there are only 213 of such curves when the genus is two and found equations for each one. For the genus greater than two case we found only 75 of such curves. Numerical evidences suggest us that these curves are the set of all new modular hyperelliptic curves.