Quasi exponential decay of a finite difference space discretization of the 1-d wave equation by pointwise interior stabilization

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Outline of the talk

- Continuous problem
- Semi-discretization
- Energy
  - Energy decay
  - Energy decay to 0
- Non exponential decay of the discrete energy
- Filtering technique
  - Interior observability
  - A preliminary estimate
  - "Quasi" exponential decay

Continuous problem

$$\begin{cases} y_{tt} - y_{xx} = 0 & 0 < x < 1,\, t > 0, \\ y(0,\, t) = 0,\, y_x(1,\, t) = 0 & t > 0, \\ y(\xi_-,\, t) = y(\xi_+,\, t) & t > 0, \\ y_x(\xi_-,\, t) - y_x(\xi_+,\, t) = -\alpha y_t(\xi,\, t) & t > 0, \\ y(t=0) = y^{(0)},\, y_x(t=0) = y^{(1)} & 0 < x < 1. \end{cases} \tag{1}$$

We define

$$V := \{y \in H^1(0,\, 1)\,;\, y(0) = 0\}$$

Proposition

For all $(y^{(0)}, y^{(1)}) \in V \times L^2(0,\, 1)$ *and for all* $\alpha > 0$, *there exists a unique solution*

$$y \in C((0,\, T),\, V) \cap C^1((0,\, T),\, L^2(0,\, 1)).$$

Energy

The energy of the solution of system (1) is given by

$$E(t) = \frac{1}{2}\int_0^1 (\left|y_t(x,\,t)\right|^2 + \left|y_x(x,\,t)\right|^2)dx$$

and obeys the following dissipation law :

$$\frac{dE(t)}{dt} = -\alpha\left|y_t(\xi,\,t)\right|^2. \tag{2}$$

This implies that the energy is decreasing. Moreover, $\lim_{t\to\infty} E(t) = 0$ for any initial data in $V \times L^2(0,\,1)$ if and only if

$$\xi \neq \frac{2p}{2q+1},\ \forall p, q \in \mathbb{N}.$$

Exponential decay

The exponential decay property of the solution of (1) is equivalent to an observability estimate for the corresponding conservative system

$$\begin{cases} \varphi_{tt} - \varphi_{xx} = 0 & 0 < x < 1,\, t > 0, \\ \varphi(0,\, t) = 0,\, \varphi_x(1,\, t) = 0 & t > 0, \\ \varphi(t=0) = y^{(0)},\, \varphi_x(t=0) = y^{(1)} & 0 < x < 1, \end{cases} . \tag{3}$$

In this case, the observability estimate holds if and only if $\xi = \frac{p}{q}$, where p is odd, and therefore, the system (1) is exponentially stable in the energy space [*Ammari-Henrot-Tucsnak 2001*].

Semi-discretization

Let $N \in \mathbb{N}$ and $h = \frac{1}{N+1}$ and consider the subdivision of $(0, 1)$ given by

$$0 = x_0 < ... < x_{j-1} < x_j = jh < x_{j+1} < ... < x_{N+1} = 1,$$

i.e. $x_j = jh$ for all $j = 0, ..., N+1$.

We fix $j_N \in \mathbb{N} \cap (0, N+1)$ such that $x_{j_N} \to \xi$ when $N \to \infty$.

Finite difference scheme

The finite-difference space semi-discretization of system (1) that we consider is the following

$$\left\{\begin{array}{ll} y_j'' - \frac{y_{j+1}-2y_j+y_{j-1}}{h^2} = 0 & t>0,\ j=1,...,N,\ j\neq j_N, \\ y_0=0,\ y_{N+1}-y_N=0 & t>0, \\ \frac{y_{j_N+1}-2y_{j_N}+y_{j_N-1}}{h} = \alpha y_{j_N}' & t>0, \\ y_j(t=0)=y_j^{(0)}, & \\ y_j'(t=0)=y_j^{(1)} & j=1,...,N \end{array}\right. . \quad (4)$$

We set $y_h=(y_j)_j$, $y_h^{(0)}=(y_j^{(0)})_j$ and $y_h^{(1)}=(y_j^{(1)})_j$.

Energy decay

We define the energy as

$$E_h(t) = \frac{h}{2} \sum_{j=0,\, j\neq j_N}^{N} \left|y_j'(t)\right|^2 + \frac{h}{2} \sum_{j=0}^{N} \left|\frac{y_{j+1}(t) - y_j(t)}{h}\right|^2, \qquad (5)$$

Proposition

The energy is decreasing with respect to t and

$$E_h'(t) = -\alpha (y_{j_N}'(t))^2.$$

Conservative system

We consider the finite-difference space semi-discretization of conservative system (3) :

$$\begin{cases} u_j'' - \frac{u_{j+1}-2u_j+u_{j-1}}{h^2} = 0 & t>0,\, j=1,...,\, N \\ u_0 = 0,\, u_{N+1} - u_N = 0 & t>0 \\ u_j(t=0) = y_j^{(0)},\, u_j'(t=0) = y_j^{(1)} & j=1,...,\, N \end{cases} . \qquad (6)$$

The system (6) can be rewritten in the following simplified form :

$$U_h'' + A_h U_h = 0,\, t>0$$

where $U_h = (u_1, ..., u_N)$ and $A_h = \frac{1}{h^2}\, tridiag(-1,\, 2,\, -1)$.

The eigenvectors of the matrix A_h satisfy the eigenvalue system

$$\begin{cases} -\frac{\varphi_{j+1}-2\varphi_j+\varphi_{j-1}}{h^2} = \lambda\varphi_j & j = 1, ..., N \\ \varphi_0 = 0,\ \varphi_{N+1} - \varphi_N = 0 & \end{cases} . \tag{7}$$

From [*Isaacson-Keller 1966*] we have

$$\varphi_j^{k,\,h} = \sin\left(\frac{(2k+1)\pi j h}{2-h}\right),\ j = 0,\ 1, ...,\ N$$

$$\lambda_{k,\,h} = \frac{4}{h^2}\,\sin^2\left(\frac{(2k+1)\pi h}{2(2-h)}\right),\ k = 0,\ 1, ...,\ N-1. \tag{8}$$

Energy of conservative system

The energy of the conservative system (6) is given by

$$E_{j_N}(u_h,\, t) = \frac{h}{2}\sum_{j=0}^{N}\left(\left|u_j'(t)\right|^2 + \left|\frac{u_{j+1}(t) - u_j(t)}{h}\right|^2\right). \qquad (9)$$

Obviously, this energy $E_{j_N}(u_h,\, .)$ is constant.
We now introduce a new energy $\tilde{E}_h$ for (6) by

$$\tilde{E}_h(u_h,\, t) = \frac{h}{2}\sum_{j=0}^{N}\left|\frac{u_{j+1}'(t) - u_j'(t)}{h}\right|^2 + \frac{h}{2}\sum_{j=1}^{N}\left|(A_h u_h)_j(t)\right|^2. \qquad (10)$$

This new energy $\tilde{E}_h$ of the conservative system (6) is constant.

Energy decay to 0

Proposition

$\lim_{t\to\infty} E_h(t) = 0$ *if and only if*

$$j_N h \neq \frac{(2-h)l}{2k+1}, \; \forall k = 0, ..., N-1, \; l \in \mathbb{N}.$$

Proof :

$\Leftarrow$: La Salle's invariance principle.

$\Rightarrow$: If $\exists k \in \{0, ..., N-1\}$ such that $j_N h = \frac{(2-h)l}{2k+1}$, then $\varphi^{k,\,h}_{j_N} = 0$ and

$$y_h(t) = \varphi^{k,\,h} \cos(\lambda_{k,\,h} t)$$

is solution of (4) with a constant energy.

A splitting

We split up y_h solution of (4) : $y_h = u_h + w_h$, where $u_h = (u_j)_j$ is solution of (6) and $w_h = (w_j)_j$ solves

$$\begin{cases} w_j'' - \frac{w_{j+1} - 2w_j + w_{j-1}}{h^2} = 0 & t > 0,\, j = 1, ..., N,\, j \neq j_N \\ w_0 = 0,\, w_{N+1} - w_N = 0 & t > 0 \\ \frac{w_{j_N+1} - 2w_{j_N} + w_{j_N-1}}{h} = \alpha y'_{j_N} - h(A_h u_h)_{j_N} & t > 0 \\ w_j(t=0) = 0,\, w_j'(t=0) = 0 & j = 1, ..., N \end{cases} . \tag{11}$$

The energy

$$E_h(w_h, t) = \frac{h}{2} \sum_{j=0,\, j \neq j_N}^{N} \left| w_j'(t) \right|^2 + \frac{h}{2} \sum_{j=0}^{N} \left| \frac{w_{j+1}(t) - w_j(t)}{h} \right|^2$$

verifies

$$E_h'(w_h, t) = -w'_{j_N} (\alpha y'_{j_N} - h(A_h u_h)_{j_N}). \tag{12}$$

We fix $\xi = \frac{p}{q}$ where p is odd and a sequence $j_N \in \mathbb{N}$ such that $x_{j_N} = j_N h \to \xi = \frac{p}{q}$, when $h \to 0$.

Lemma

If the decay of E_h is exponential, then $\exists T > 0$ and $C > 0$ such that $\forall y_h^{(0)}, y_h^{(1)} \in \mathbb{R}^N$, one has

$$E_{j_N}(u_h, 0) \leq C \int_0^T \left|u'_{j_N}\right|^2 dt + \frac{h}{2} \int_0^h \left|(A_h u_h)_{j_N}\right|^2 dt + Ch^2 \int_0^T \left|(A_h u_h)_{j_N}\right|^2 dt$$

where u_h is solution of (6) and $E_{j_N}(u_h, .)$ is defined by (9).

Non exponential decay (2)

Lemma

Assume that

- *N is a multiple of q*
- *$j_N = N\frac{p}{q}$, where p is odd (and so $x_{j_N} \to \frac{p}{q}$).*

Then $\forall T > 0$, $\exists C(T) > 0$ and initial data such that the solution u_h of (6) with these initial data satisfies

$$E_{j_N}(u_h, 0) \geq \frac{C(T)}{h^2}\Big(\int_0^T \left|u'_{j_N}(t)\right|^2 dt + \frac{h}{2}\int_0^h \left|(A_h u_h)_{j_N}\right|^2 dt + Ch^2 \int_0^T \left|(A_h u_h)_{j_N}\right|^2 dt\Big).$$

Theorem

Assume that

- N *is a multiple of* q
- $j_N = N\frac{p}{q}$, *where* p *is odd (and so* $x_{j_N} \to \frac{p}{q}$*).*

Then the decay of E_h *to zero is not uniformly exponential with respect to* h*. More precisely there do not exist positive constants* M *and* ω *which are independant of* h *such that for all* $h > 0$ *and* $y_h^{(0)}$ *and* $y_h^{(1)}$ *in* $\mathbb{R}^N$,

$$E_h(t) \le Me^{-\omega t}E_h(0), \, \forall t \ge 0. \tag{13}$$

Filtering technique

Every solution of (6) can be developped in Fourier series as follow

$$u_h(t) = \sum_{k=0}^{N-1} [a_k \cos(\sqrt{\lambda_{k,\,h}}t) + \frac{b_k}{\sqrt{\lambda_{k,\,h}}} \sin(\sqrt{\lambda_{k,\,h}}t)]\varphi^{k,\,h}$$

where $a_k,\ b_k \in \mathbb{R}$, $k = 0, ..., N-1$. Introduce

$$C_h(\gamma) := \left\{ u_h = \sum_{\lambda_{k,\,h} \leq \frac{\gamma}{h^2}} a_k \varphi^{k,\,h} \quad \text{with } a_k \in \mathbb{R} \right\}.$$

The gap condition

Lemma

Assume that $\gamma = 4\sin^2(\frac{\pi\epsilon}{2})$ *for some* $0 \leq \epsilon < 1$. *Then*

$$\sqrt{\lambda_{k,\,h}} - \sqrt{\lambda_{k-1,\,h}} \geq \pi\cos(\frac{\pi\epsilon}{2})$$

for all eigenvalues in the range $\lambda h^2 \leq \gamma$.

We fix $\xi = \frac{p}{q}$ where p is odd and take the sequence $j_N \in \mathbb{N}$ such that $x_{j_N} = j_N h \to \xi = \frac{p}{q}$, when $h \to 0$, defined by $j_N = \left[\frac{p(2N+1)}{2q}\right] \in \mathbb{N}$, where $[x]$ means the integral part of x.

Proposition 1

Assume that $\epsilon \in (0, \frac{\pi}{2})$ is small enough such that $\alpha > \tan(\frac{\pi\epsilon}{2})$, where α verifies

$$\left|\sin((k+\frac{1}{2})\pi\frac{p}{q})\right| > \alpha, \forall k \in \mathbb{N}.$$

Then there exist $T = T(\gamma) > 2$ and $C = C(\gamma, T) > 0$ such that for every solution of (4) in the class $C_h(\gamma)$

$$(T-2)E_{j_N}(u_h, 0) \leq C \int_0^T \left|u'_{j_N}(t)\right|^2 dt$$

uniformly as $h \to 0$.

Proof: based on a boundary observability estimate from [*Tcheugoué Tébou-Zuazua 2007*] and Ingham's inequality.

A preliminary estimate

Proposition

There exist $T > 0$ and $C(T) > 0$ such that the solution w_h of (11) verifies

$$\int_0^T (w'_{j_N})^2 dt \leq C(T)[\int_0^{2T} (y'_{j_N})^2 dt + h\tilde{E}_h(u_h, 0) + h\tilde{F}_h(u_h, 0)],$$

where y_h (respectively u_h) is solution of (4)(respectively (6)), $\tilde{E}_h$ is defined by (10) and $\tilde{F}_h$ given by

$$\tilde{F}_h(u_h, t) = \frac{h}{2} \sum_{j=0}^{N} \left| \frac{u''_{j+1}(t) - u''_j(t)}{h} \right|^2 + \frac{h}{2} \sum_{j=1}^{N} |(A_h u'_h)_j(t)|^2$$

is obtained by substituting u_h by u'_h in the energy $\tilde{E}_h$.

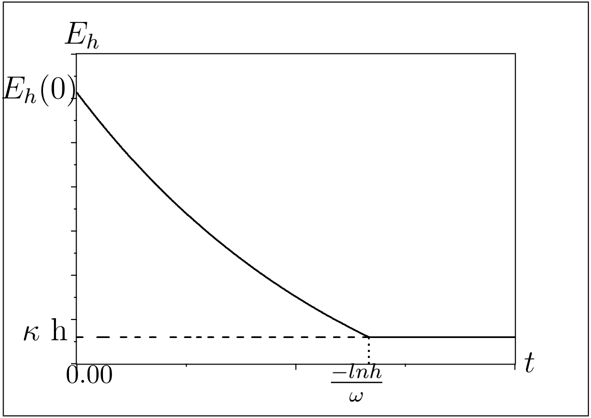
"Quasi" exponential decay

Theorem

Under the assumptions of Proposition 1, there exist $K > 0$, $\omega > 0$ *and* $C > 0$ *such that any solution of (4) with initial data in* $C_h(\gamma)$ *verifies*

$$E_h(y_h, t) \leq Ke^{-\omega t} E_h(y_h, 0) + Ch(\tilde{E}_h(u_h, 0) + \tilde{F}_h(u_h, 0)).$$

An illustration



Equivalence

Proposition

Let the assumptions of Proposition 1 be satisfied, the following assertions are equivalent :
(i) There exist $K > 0$, $\omega > 0$ *and* $C > 0$ *such that every solution of (4) with initial data in* $C_h(\gamma)$ *verifies*

$$E_h(y_h, t) \leq K e^{-\omega t} E_h(y_h, 0) + Ch(\tilde{E}_h(u_h, 0) + \tilde{F}_h(u_h, 0)).$$

(ii) There exist positive constants T_0 *and* C_0 *such that every solution of (6) satisfies*

$$E_{j_N}(u_h, t) \leq C_0 \int_0^{T_0} \left| u'_{j_N} \right| dt + C_0 h(\tilde{E}_h(u_h, 0) + \tilde{F}_h(u_h, 0)).$$

Conclusion and open problems

Conclusion

- Non uniform exponential decay of the discrete energy
- Filtering technique allows to restore a quasi exponential decay

Open problems

- Uniform polynomial decay rates
- Full discretizations
- Discretization on networks