Remarks on Carleman estimates for the heat equation with an inverse-square potentials.

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Universidad Autónoma, December 2005

Outline of the talk

1. Introduction
2. Related works and known results
3. Personal contribution
4. Open problems

Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, with $N \geq 3$, and $\omega \subset \Omega$.

The equation

We consider the null controllability problem of the heat equation with an inverse square potential.

$$\begin{cases} \partial_t u - \Delta u - \dfrac{\mu}{|x|^2} u = v\chi_\omega, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1}$$

Precise statement of the problem

Problem

Given any $u_0 \in L^2(\Omega)$, can we find a control $v \in L^2(\omega \times (0,T))$ such that the solution u of (1) satisfies

$$u(T, v) = 0 \quad ? \tag{2}$$

Remarks on the Cauchy problem

Careful ! If $\mu > \mu^*(N)$ and if $u_0 \geq 0$, then *no solution exists*[*], *even locally in time !*

The Cauchy problem[†] is well-posed if μ is not too large, namely

$$\mu \leq \mu^*(N) = \left(\frac{N-2}{2}\right)^2. \tag{3}$$

Hardy

Strongly related to the Hardy inequality

$$\forall u \in H_0^1(\Omega), \quad \mu^*(N) \int_\Omega \frac{u^2}{|x|^2} \,\mathrm{dx} \leq \int_\Omega |\nabla u|^2 \,\mathrm{dx}. \tag{4}$$

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Remarks on the Cauchy problem

- Functional spaces:

$$u \in H_0^1(\Omega) \Longrightarrow \frac{u}{|x|^2} \in H^{-1}.$$

- Positivity of the operator

$$-\Delta - \frac{\mu^*(N)}{|x|^2}.$$

- Hardy not attained ! ⟶ A natural norm is

$$\|u\|^2 = \int |\nabla u|^2 - \mu^*(N) \int |u|^2$$

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Previous works

- **On the heat equations:**
  Fursikov & Imanuvilov, *Controllability of evolution equations* (1996),
  Lebeau & Robbiano, *Contrôle exact de l'équation de la chaleur* (1995),
  Fernandez-Cara & Zuazua, *Null and approximate controllability for weakly blowing up semilinear heat equation* (2000),
- **Heat equations with singular coefficients:**
  Martinez & Vancostenoble, *Carleman estimates for one-dimensional degenerate heat equations* (2007)
  Vancostenoble & Zuazua, *Null-controllability for the heat equation with singular inverse-square potentials* (2007).

Known results

Heat equation with a potential

Controllability of

$$\begin{cases} \partial_t u - \Delta u - a(x) u = v \chi_\omega, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Ok[‡] if

$$a \in L^p(\Omega), \quad p > 2N/3.$$

Here,

$$\frac{1}{|x|^2} \in L^{N/2-\epsilon}(\Omega), \quad \varepsilon > 0 \quad !$$

[‡] Imanuvilov & Yamamoto (2003)

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Known results

Conjecture and open problem

- We do not expect controllability to hold for

$$a \in L^p, \quad p < N/2.$$

- We do not know what happens when

$$a \in L^p(\Omega), \quad p \in (N/2, 2N/3] \quad !$$

Anyway, with the inverse square potential, we cannot apply the existing theory !

Known results

Controllability of the heat equation with a singular inverse-square potential holds[§] under some geometric assumptions on the control region ω, namely :

Assumption

ω contains an annular set which surrrounds the singularity.

Key point of the proof: Use spherical harmonics to work with radial equations near the singularity, and a Carleman estimate on a 1-d heat equation with a singular potential.

Our goal: Remove the geometrical assumption.

From now, we assume $\bar{\omega} \cap \bar{B}(O, 1) = \phi$. (*scaling argument*)

[§] Vancostenoble & Zuazua, 2007

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Controllability result

Theorem

Controllability holds for (1) for $\mu \leq \mu^*(N)$ without any geometric assumption !

Difficulties :

- We cannot use spherical harmonics.
- Choosing a weight conveniently in a Carleman estimate.

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Sketch of the proof

Step 1

Use HUM, and hence rather study the observability of the adjoint system, that is the inequality

$$\int_{\Omega} |w(x,0)|^2 \,\mathrm{dx} \leq C \iint_{\omega\times(0,T)} |w(x,t)|^2 \,\mathrm{dxdt} \tag{5}$$

for any solution of

$$\begin{cases} \partial_t w + \Delta w + \dfrac{\mu}{|x|^2} w = 0, & (x,t) \in \Omega \times (0,T), \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ w(x,T) = w_T(x), & x \in \Omega. \end{cases} \tag{6}$$

Sketch of the proof

Step 2

Derive a Carleman estimate to prove the observability inequality (5).

Let ψ enjoying the following properties:

$$\begin{cases} \psi(x) = \ln(|x|), & x \in B(O,1), \\ \psi(x) = 0, & x \in \partial\Omega, \\ \psi(x) > 0, & x \in \Omega \backslash \bar{B}(O,1), \\ |\nabla\psi(x)| \geq \delta > 0, & x \in \bar{\Omega}\backslash\omega, \end{cases} \tag{7}$$

define $\sigma(t,x)$ the weight function

$$\sigma(t,x) = s\theta(t)\Big(e^{2\lambda \sup\psi} - \frac{1}{2}|x|^2 - \phi(x)\Big), \tag{8}$$

where s and λ are large coefficients, and θ and ϕ are

$$\theta(t) = \left(\frac{1}{t(T-t)}\right)^3, \qquad \phi(x) = e^{\lambda\psi(x)}.$$

Carleman estimate

Theorem

$\exists K > 0, \exists \lambda_0 > 0, \forall \lambda \geq \lambda_0, \exists s_0(\lambda), \forall s \geq s_0$, any w solution of (6) satisfies

$$s\lambda^2 \iint_{\Omega\backslash B(O,1)} \theta\phi e^{-2\sigma}|\nabla w|^2 + s^3\lambda^4 \iint_{\Omega\backslash B(O,1)} \theta^3\phi^3 e^{-2\sigma}|w|^2$$
$$+ s \iint_{\Omega\times(0,T)} \theta e^{-2\sigma}\frac{|w|^2}{|x|} + s^3 \iint_{\Omega\times(0,T)} \theta^3 e^{-2\sigma}|x|^2|w|^2$$
$$\leq K\left(s\lambda^2 \iint_{\omega\times(0,T)} \theta\phi e^{-2\sigma}|\nabla w|^2 + s^3\lambda^4 \iint_{\omega\times(0,T)} \theta^3\phi^3 e^{-2\sigma}|w|^2\right)$$

Comments

- The weight function looks like $\theta(t)(C - |x|^2)$ in the unit ball as the weight proposed in Vancostenoble & Zuazua to study the radial singular equations.
- Outside the ball, the weight functions looks like the weight function proposed by Fursikov & Imanuvilov to study the observability of the heat equation.
- If w is a solution of (6) with a source term f, the same inequality holds provided we add

$$\iint_{\Omega\times(0,T)} e^{-2\sigma}|f|^2$$

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Comments

- In the proof, we use the following Hardy inequality:

$$\mu^*(N)\int_\Omega \frac{|w|^2}{|x|^2} + \int_\Omega \frac{|w|^2}{|x|} \leq \int_\Omega |\nabla w|^2 + C\int_\Omega |w|^2.$$

- If $\mu < \mu^*(N)$, one can add in the left-hand side the following term

$$s(\mu^*(N)-\mu)\iint_{\Omega\times(0,T)} \theta e^{-2\sigma}\frac{|w|^2}{|x|^2}.$$

- For any $\gamma < 2$, replacing θ by $(t(T-t))^{-1-2/\gamma}$, one can obtain in the right hand-side

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To the observability

Step 3

Using Cacciopoli's inequality, one easily obtain

$$\iint_{\Omega\times(\frac{T}{4},\frac{3T}{4})} |w(x,t)|^2 \leq C \iint_{\omega\times(0,T)} |w(x,t)|^2.$$

Due to the dissipation properties of (6), this implies the observability inequality

$$\int_{\Omega} |w(x,0)|^2 \,\mathrm{dx} \leq C \iint_{\omega\times(0,T)} |w(x,t)|^2 \,\mathrm{dxdt}.$$

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Further comments

Other singular potentials

Using the same Carleman estimate, one can prove the controllability of the heat equation with more general potentials

$$\begin{cases} \partial_t u - \Delta u - \frac{\mu}{|x|^2} u + \frac{m}{|x|^\gamma} u = v\chi_\omega, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

with $\gamma < 2$, and still $\mu \leq \mu^*(N)$.

Multipolar singular potentials

Multipolar case

Consider

$$\begin{cases} \partial_t u - \Delta u - \sum_i \frac{\mu_i}{|x - x_i|^2} u = v\chi_\omega, & (x,t) \in \Omega \times (0,T), \\ u = 0, \quad \text{on } \partial\Omega \end{cases} \tag{9}$$

where $\mu_i \leq \mu^*(N)$ for each i, and the set $\{x_i\}$ is finite.

Result

Controllability for system (9) holds !

Without loss of generality, we assume (*scaling*)

$$\inf\{|x_i - x_j|\} \geq 2, \qquad d(x_i, \partial\Omega) \geq 2.$$

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Multipolar Carleman estimate

The weight function

The following choice gives a "nice" Carleman inequality for (9)

$$\sigma(t,x) = s\theta\left(e^{2\lambda \sup \psi} - \frac{1}{2}\sum_i |x-x_i|^2\gamma(x-x_i) - e^{\lambda\psi(x)}\right),$$

where ψ satisfies

$$\begin{cases} \psi(x) = \ln(|x-x_i|), & x \in B(x_i,1), \\ \psi(x) = 0, & x \in \partial\Omega, \\ \psi(x) > 0, & x \in \Omega \backslash \left(\cup_i \bar{B}(x_i,1)\right), \\ |\nabla\psi| \geq \delta > 0, & x \in \Omega\backslash\omega \end{cases}$$

and γ is such that

$$\gamma(x) = 1, \quad |x| \leq 1, \qquad \gamma(x) = 0, \quad |x| \geq 2.$$

Remarks

From the Carleman estimate, any solution of (9) satisfies

$$\iint_{\Omega\times(\frac{T}{4},\frac{3T}{4})} |w(x,t)|^2 \leq C \iint_{\omega\times(0,T)} |w(x,t)|^2.$$

But the system is not dissipative anymore !
Indeed, the quadratic form

$$Q(w) = \int_\Omega |\nabla w|^2 - \sum_i \int_\Omega \frac{\mu_i}{|x-x_i|^2}|w|^2$$

is not positive !

However, it is bounded from below, and this is enough to conclude!

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$\mu > \mu^*(N)$

Known results

- Baras & Goldstein (1984): No solution for positive initial data, even localy in time!
- Vazquez & Zuazua (2000): Well-posed Cauchy problem in the ball when filtering.
- Vancostenoble & Zuazua (2007): Controllability in the ball for filtered initial data with a control on an annular set.

Problem

Controllability without filtering ? In any set ?

Non linear parabolic equations

The operator

$$-\Delta - \frac{\mu}{|x|^2}$$

appears as the linearization of several nonlinear elliptic problems[¶]

$$\begin{cases} -\Delta u_1 = \lambda_1 e^{u_1}, \quad -\Delta u_2 = \lambda_2 (1+u_2)^p \quad \text{in } B(O,1) \\ u = 0 \quad \text{on } S(0,1) \end{cases}$$

Problem

Can we control the corresponding nonlinear parabolic equation around the stationary state u_1, (respectively u_2)?

[¶] Brézis & Vazquez, Blow-up solutions of some nonlinear elliptic equations, 1997

Wave equation with an inverse-square potential

Wave equation

Under which geometrical assumption can we control the following equation ?

$$\begin{cases} \partial_{tt}^2 u - \Delta u - \dfrac{\mu}{|x|^2} u = v\chi_\omega, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Thanks

Thank you for your attention !

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