Prove, directly, without using calculus of variations, that the 1-dimensional Euler-Lagrange equations are invariant by an arbitrary change of coordinates.

In other words, if $L = L(x, \dot{x})$, you have to prove that

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \quad \Rightarrow \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) = \frac{\partial L}{\partial y}$$

for a change of variables $y = y(x)$.

I emphasize that the proof has to be direct, using the chain rule and basic calculus a number of times. For instance,

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial L}{\partial \dot{y}} \frac{\partial \dot{y}}{\partial x}.$$ 

Note that in this expression, we would usually write $y'(x)$ instead of $\partial y/\partial x$. 