Prove, directly, that \( d\omega \) is well-defined, i.e., that it does not depend on the choice of the coordinate chart.

Here “directly” means that we can only use calculus and the very definition:

Recall that for

\[
\omega = \sum_{i_1 < i_2 < \ldots < i_k} f_{i_1i_2\ldots i_k} \, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k},
\]

we define

\[
d\omega = \sum_j \sum_{i_1 < i_2 < \ldots < i_k} \frac{\partial f_{i_1i_2\ldots i_k}}{\partial x^j} \, dx^j \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}
\]

If it is of any help, you can assume that the underlying manifold is \( \mathbb{R}^n \).