Compute the induced metric (by the usual $\mathbb{R}^3$ metric) on the sphere $S^2$ when we use the coordinate chart $(S^2 - \{ N \}, p_N)$ where $p_N$ the stereographic projection from the north pole $N$.

Solution. The inverse of the stereographic chart is

$$(x, y) \mapsto (X, Y, Z) = \left( \frac{2x}{D}, \frac{2y}{D}, 1 - \frac{2}{D} \right) \quad \text{with} \quad D = 1 + x^2 + y^2.$$ 

Then

$$dX = \frac{2D - 4x^2}{D^2} dx - \frac{4xy}{D^2} dy, \quad dY = \frac{2D - 4y^2}{D^2} dx - \frac{4xy}{D^2} dy, \quad dZ = \frac{4x}{D^2} dx + \frac{4y}{D^2} dy$$

and the induced metric is

$$\left( \frac{2D - 4x^2}{D^2} dx - \frac{4xy}{D^2} dy \right)^2 + \left( - \frac{4xy}{D^2} dy + \frac{2D - 4y^2}{D^2} dx \right)^2 + \left( \frac{4x}{D^2} dx + \frac{4y}{D^2} dy \right)^2$$

where we are using the classic notations (the squares really mean tensor products with itself).

The coefficient of $(dx)^2$ is

$$\left( \frac{2D - 4x^2}{D^2} \right)^2 + \left( \frac{4xy}{D^2} \right)^2 + \left( \frac{4x}{D^2} \right)^2 = \frac{4D^2 - 16x^2 D + 16x^2(x^2 + y^2 + 1)}{D^4} = \frac{4}{D^2}.$$ 

By the symmetry $x \leftrightarrow y$ this is also the coefficient of $(dy)^2$. There are not cross terms $dxdy$ because

$$-2(2D - 4x^2)4xy - 8xy(2D - 4y^2) + 32xy = 8xy(-2D + 4x^2 - 2D + 4y^2 + 4) = 0.$$ 

Then we obtain finally,

$$4D^{-2}((dx)^2 + (dy)^2) = 4(1 + x^2 + y^2)^{-2}((dx)^2 + (dy)^2).$$