Post-Newtonian approximations

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January 25, 2016
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Einstein’s double prediction

A ray of light grazing the Sun suffers a deflection of about 0.83”

(Annalen der Physik 35: 898–908, 1911)

Luckily the experiment was not done (pre-war crisis, weather) because...
A ray of light grazing the Sun suffers a deflection of about 0.83”

(Annalen der Physik 35: 898–908, 1911)

Luckily the experiment was not done (pre-war crisis, weather) because...

A ray of light grazing the Sun suffers a deflection of about 1.7”

Actually the first prediction was done by J.G. von Soldner (more than a century before Einstein!) just assuming that light is affected by Newtonian gravitation as any particle.

\[ \alpha \sim \frac{2GM}{R_0} \]

von Soldner 1804 → \( \alpha \sim 2GM/R_0 \)

Einstein 1911 → \( \alpha \sim 2GM/R_0 \)

Einstein 1916 → \( \alpha \sim 4GM/R_0 \)
Newtonian vs Post-Newtonian

Looking into retrospective, we can interpret today this double prediction saying that in the first prediction the metric of space-time was approximated by

$$(1 - 2U)dt^2 - (dx^2 + dy^2 + dz^2) \quad \text{with} \quad U = \frac{GM}{r}$$

and in the second by

$$(1 - 2U + 2U^2)dt^2 - (1 + 2U)(dx^2 + dy^2 + dz^2).$$
Looking into retrospective, we can interpret today this double prediction saying that in the first prediction the metric of space-time was approximated by

\[(1 - 2\mathcal{U})dt^2 - (dx^2 + dy^2 + dz^2)\]

with \[\mathcal{U} = \frac{GM}{r}\]

and in the second by

\[(1 - 2\mathcal{U} + 2\mathcal{U}^2)dt^2 - (1 + 2\mathcal{U})(dx^2 + dy^2 + dz^2).\]

Both are approximations to the actual (Schwarzschild’s) metric but the first one does not go beyond the Newtonian (von Solder’s) results and the second, more precise, is post-Newtonian. It doubles the prediction although \(\mathcal{U}\) is small in non-relativistic units.
Einstein’s double prediction

**Newtonian vs Post-Newtonian**

The PPN formalism

The classical tests

---

**General relativity**

\[
\frac{Du}{d\tau} = 0
\]

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\]

\[
G \to 0
\]

(relativistic kinematics)

**Newtonian theory**

\[
\frac{d\vec{v}}{dt} = -\nabla \Phi
\]

\[
\frac{d\vec{v}}{dt} = 0
\]

\[
G \to 0
\]

(Newtonian kinematics)

---

PN approximations
General relativity

\[ R_{\alpha\beta} = 0 \]

\[ \frac{Du}{d\tau} = 0 \]

\[ d\vec{v} \frac{dt}{d\tau} = -\nabla \Phi \]

(relativistic kinematics)

Newtonian theory

geod. deviat.

\[ \Delta \Phi = 0 \]

\[ \frac{dv}{dt} = -\nabla \Phi \]

\[ \frac{du}{d\tau} \rightarrow 0 \]

\[ \frac{d\vec{v}}{dt} \rightarrow 0 \]

\[ G \rightarrow 0 \]

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\[ \frac{d\vec{v}}{dt} = 0 \]

(Newtonian kinematics)
Before doing any approximation there are two questions to answer:

1. How far should we approximate the metric?
2. What kind of quantities should we use in the approximations?
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1. How far should we approximate the metric?
2. What kind of quantities should we use in the approximations?

We work under the assumption of weak field and slow motion of the gravitational sources. We want to employ a nearly global Minkowskian coordinate system \((t, x, y, z)\).

The usual special relativistic action for a particle is

\[
S = -m \int \sqrt{g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu}.
\]
In terms of \( v^i = dx^i / dt \)

\[
S = -m \int L \, dt \quad \text{with} \quad L = \sqrt{g_{00} + 2g_{0j}v^j + g_{jk}v^jv^k}.
\]

The Newtonian limit is \( g_{00} = 1 - 2U, \quad g_{0j} = 0, \quad g_{ij} = -\delta_{ij}, \) hence

\[
L = \sqrt{1 - 2U - v^2}.
\]

Therefore the Newtonian limit corresponds to terms \( O[2] \) in \( L \), where \( O[n] \) means \( O(v^n) \). Terms \( O[3] \) do not appear in realistic models (loss of energy) and we skip to \( O[4] \) that corresponds to

\[
g_{00} = 1 - 2U + O[4], \quad g_{0j} = O[3], \quad g_{ij} = -\delta_{ij} + O[2].
\]

The approximations are done in terms of the \textit{matter variables} \( \rho, \Pi \) (internal energy), \( v \) and \( p \) (pressure), with \( p/\rho = \Pi = O[2] \).
Each metric theory has its own post-Newtonian approximations and the way of finding them can be long and difficult and completely different in each case.

Although these potentially big differences, the final expressions keep the same flavor.

The **parametrized post-Newtonian formalism** (PPN) is a comprehensive family of post-Newtonian approximations depending on parameters that can be adjusted to represent, in practice, every meaningful metric theory of gravitation.
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1. Provide a framework to compare theories.
2. Standardize the way of presenting the experimental results.
3. (?) Compute general relativity corrections in astrophysics.

The first point is the main one. Theories sharing the same PPN parameters could only be distinguished with post-post-Newtonian experiments that are commonly out of reach with current methods because the PPN formalism mimics relativistic theories with high accuracy.
The modern formalism, uses ten parameters $\gamma$, $\beta$, $\xi$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $\zeta_1$, $\zeta_2$, $\zeta_3$ and $\zeta_4$ having physical meaning, and the metric

\[
\begin{align*}
g_{00} &= 1 - 2U + \lambda_1 U^2 + \lambda_2 \Phi_W + \lambda_3 \Phi_1 + \lambda_4 \Phi_2 \\
&\quad + \lambda_5 \Phi_3 + \lambda_6 \Phi_4 + \lambda_7 A \\
g_{0j} &= \lambda_8 V_j + \lambda_9 W_j \\
g_{ij} &= -\delta_{ij} + \lambda_{10} U \delta_{ij}
\end{align*}
\]

where

\[
\begin{align*}
\lambda_1 &= 2\beta \\
\lambda_2 &= 2\xi \\
\lambda_3 &= -2\gamma - 2 - \alpha_3 - \zeta_1 + 2\xi \\
\lambda_4 &= -2(3\gamma - 2\beta + 1 + \zeta_2 + \xi) \\
\lambda_5 &= -2(1 + \zeta_3) \\
\lambda_6 &= -2(3\gamma + 3\zeta_4 - 2\xi) \\
\lambda_7 &= \zeta_1 - 2\xi \\
\lambda_8 &= \frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi) \\
\lambda_9 &= \frac{1}{2}(1 + \alpha_2 - \zeta_1 + 2\xi) \\
\lambda_{10} &= -2\gamma
\end{align*}
\]
Einstein’s double prediction Newtonian vs Post-Newtonian The PPN formalism The classical tests

\[
\begin{align*}
    g_{00} &= 1 - 2U + \lambda_1 U^2 + \lambda_2 \Phi_W + \lambda_3 \Phi_1 + \lambda_4 \Phi_2 \\
    &\quad + \lambda_5 \Phi_3 + \lambda_6 \Phi_4 + \lambda_7 A \\
    g_{0j} &= \lambda_8 V_j + \lambda_9 W_j \\
    g_{ij} &= -\delta_{ij} + \lambda_{10} U \delta_{ij}
\end{align*}
\]

The functions of accompanying the coefficients are the potentials of the formalism. The simplest one is the Newtonian potential

\[
U = \int \frac{\rho(\vec{y}, t)}{|\vec{x} - \vec{y}|} \, d^3\vec{y}
\]

and the most complicated (the variable \( t \) is not displayed)

\[
\Phi_W = \int \rho(\vec{y}) \rho(\vec{z}) \frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|^3} \cdot \left( \frac{\vec{y} - \vec{z}}{|\vec{x} - \vec{z}|} - \frac{\vec{x} - \vec{z}}{|\vec{y} - \vec{z}|} \right) \, d^3\vec{y} \, d^3\vec{z}.
\]
The parameter \( \xi \) that multiplies this latter potential was introduced historically to cover a gravitational theory proposed by A.N. Whitehead. The parameters \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) measure the existence of a universal rest frame. On the other hand \( \zeta_1, \zeta_2, \zeta_3 \) and \( \zeta_4 \) (and in part \( \alpha_3 \)) are related to the violation of global conservation laws.
The parameter $\xi$ that multiplies this latter potential was introduced historically to cover a gravitational theory proposed by A.N. Whitehead. The parameters $\alpha_1$, $\alpha_2$ and $\alpha_3$ measure the existence of a *universal rest frame*. On the other hand $\zeta_1$, $\zeta_2$, $\zeta_3$ and $\zeta_4$ (and in part $\alpha_3$) are related to the violation of global conservation laws.

When all of these parameters are set to zero (as in the case of general relativity) the PPN formalism reduces considerably. If also $p = \Pi = 0$ (as happen with point masses) and the gravitational sources are static, only the terms with $U$ survive. In this case, we have

$$(1 - 2U + 2\beta U^2)dt^2 - (1 + 2\gamma U)(dx^2 + dy^2 + dz^2).$$
If we do not assume static sources, we have to add to $g_{00}$ the term

$$-2(\gamma + 1) \int \frac{\rho(\vec{y})v^2(\vec{y})}{|\vec{x} - \vec{y}|} d^3\vec{y}$$

and cross terms appear

$$g_{0j} = \frac{1}{2}(4\gamma + 3) V_j + \frac{1}{2} W_j$$

where

$$V_j = \int \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|} v_j(\vec{y}) \, d^3\vec{y}$$

and

$$W_j = \int \frac{\rho(\vec{y})}{|\vec{x} - \vec{y}|} \frac{x_j - y_j}{|\vec{x} - \vec{y}|} \frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|} \cdot \vec{v}(\vec{y}) \, d^3\vec{y}.$$
For general relativity

\[ \gamma = \beta = 1, \quad \text{rest of the parameters} = 0 \]

\[ (1 - 2\mathcal{U} + 2\beta \mathcal{U}^2)\, dt^2 - (1 + 2\gamma \mathcal{U})(dx^2 + dy^2 + dz^2) \]
For general relativity

\[ \gamma = \beta = 1, \quad \text{rest of the parameters} = 0 \]

\[
(1 - 2\mathcal{U} + 2\beta\mathcal{U}^2)dt^2 - (1 + 2\gamma\mathcal{U})(dx^2 + dy^2 + dz^2)
\]

**Example:** Schwarzschild’s isotropic metric, \( r \leftrightarrow r(1 + GM/2r)^2 \),

\[
\left( \frac{1 - GM/2r}{1 + GM/2r} \right)^2 dt^2 - \left( 1 + \frac{GM}{2r} \right)^4 (dx^2 + dy^2 + dz^2)
\]

and use

\[
\left( \frac{1 - x/2}{1 + x/2} \right)^2 = 1 - 2x + 2x^2 + O(x^3), \quad \left( 1 + \frac{x}{2} \right)^4 = 1 + 2x + O(x^2)
\]

with \( x = \mathcal{U} = GM/r \).
Usually the following facts are considered the classical tests of general relativity:

(i) The gravitational red shift
(ii) The deflection of light
(iii) The perihelion shift of Mercury
The classical tests

Usually the following facts are considered the classical tests of general relativity:

(i) The gravitational red shift
(ii) The deflection of light
(iii) The perihelion shift of Mercury

Actually (i) is not a (good) test of GR because it is a consequence of the equivalence principle. It holds for any reasonable (metric) theory.

(ii) with the right deflection and (iii) are Post-Newtonian but...
L. Schiff:
The deflection of light is a consequence of the equivalence principle and special relativity.


And 8 years latter...
Deflection of light

L. Schiff:
The deflection of light is a consequence of the equivalence principle and special relativity.


And 8 years latter...

W. Rindler:
The deflection of light cannot be derived from the equivalence principle and special relativity.

In the PPN formalism the equations of motion for light are

\[
\begin{align*}
1 - 2U - \left| \frac{d\vec{x}}{dt} \right|^2 (1 + 2\gamma U) &= 0 \\
\frac{d^2 x^j}{dt^2} &= \frac{\partial U}{\partial x^j} \left( 1 + \gamma \left| \frac{d\vec{x}}{dt} \right|^2 \right) - 2 \frac{dx^j}{dt} \left( \frac{d\vec{x}}{dt} \cdot \nabla U \right) (1 + \gamma)
\end{align*}
\]

First approximation → Newtonian equation \( \frac{d^2 \vec{x}}{dt^2} = \frac{\partial U}{\partial x^j} \)

It only depends on \( \gamma \) then the deflection of light is a good method to measure this parameter.

Tracking of Cassini space probe → \( \gamma = 1 \pm 2.3 \cdot 10^{-5} \)
The PPN formalism with $\alpha_1 = \alpha_2 = \alpha_3 = \zeta_2 = 0$ or under the assumption of a negligible planet mass, gives

$$\alpha \sim \pi GM \left( \frac{1}{r_a} + \frac{1}{r_p} \right) \left( 2 + 2\gamma - \beta \right) + \frac{3\pi}{4} J_2 R^2 \left( \frac{1}{r_a} + \frac{1}{r_p} \right)^2$$

where $J_2 =$ quadrupole moment, $R =$ mean radius of the Sun.

A possible discrepancy between GR and the experimental solar oblateness found by R.H. Dicke raised a long controversy during the 60’s and early 70’s.

Mercury observations $\rightarrow |2\gamma - \beta - 1| < 3 \cdot 10^{-3}$. 
The PN approximation of GR gives a “rotation of the ellipse” of angle

$$\alpha \sim 3\pi GM \left( \frac{1}{r_a} + \frac{1}{r_p} \right)$$

per each orbital period, where $r_a$ and $r_p$ are the (distances to) the aphelion and perihelion.

**General relativity predictions**

- Mercury → 42.98” /century
- Venus → 8.63” /century
- Earth → 3.84” /century
- Mars → 1.35” /century
In many books of general relativity it is assumed a nearly circular orbit in the perihelion shift. This is not very convincing:

1. The main interest is to apply it to Mercury and its orbit is clearly eccentric.
2. In a nearly circular orbit the perihelion is too sensitive to approximations.

In the perihelion a minimum is attached. Minimum attaching points are not well localized when using flat function approximations:

- Not flat
- Flat
It is better to use that for every cubic polynomial

\[ P(x) = \epsilon x^3 - bx^2 + cx - d \quad \text{with } \epsilon, b > 0 \]

and three real roots \(0 < x_1 < x_2 < x_3\) we have the approximation

\[
\int_{x_1}^{x_2} \frac{dx}{\sqrt{P(x)}} \sim \frac{\pi}{\sqrt{b}} + \frac{3\pi}{4b^{3/2}}(x_1 + x_2)\epsilon
\]

which is valid whenever \(\epsilon x_2 / b\) is small.

In the application, \(\epsilon = 2GM\) for GR and \(\epsilon = 2AGM\) (with \(A\) essentially the quadrupole moment over the square of the angular momentum of the planet) for the Newtonian theory.
Appendix: From Newton to Einstein

Parametrize the trajectories of dust of Newtonian test particles
$$\alpha(t, s) = (x(t, s), y(t, s), z(t, s))$$
and consider the vector fields that points to the “adjacent” particle

$$\alpha(t, s) = (x(t, s), y(t, s), z(t, s))$$
$$\vec{V} = \frac{\partial \alpha}{\partial t}(t, 0).$$
Appendix: From Newton to Einstein

The, so to speak, relative acceleration is

\[ \frac{d^2 V^i}{dt^2} = -\delta^{ij} \frac{\partial}{\partial s} \bigg|_{s=0} \partial_j \Phi(\alpha(t, s)) = -\delta^{ij} (\partial_k \partial_j \Phi) V^k. \]

Defining \( A^i_k = \delta^{ij} \partial_k \partial_j \Phi \) we can re-write this as

\[ \frac{d^2 V^i}{dt^2} + A^i_k V^k = 0 \quad \text{with} \quad A^i_i = 4\pi G \rho. \]

Comparing with the geodesic deviation formula

\[ \frac{D^2 V^\alpha}{dt^2} + A^\alpha_\gamma V^\gamma \quad \text{where} \quad A^\alpha_\gamma = R^\alpha_{\beta \gamma \delta} \dot{x}^\beta \dot{x}^\delta \]

one can infer (imagine) Einstein’s field equations.