Spectral, combinatorial and analytic methods in some problems in number theory

Dulcinea Raboso

- Ph.D defense -

Advisor: Fernando Chamizo

July 8th, 2014
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   - Exotic approximate identities and Maass forms

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     - Lattice points in the 3-dimensional torus
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**Non-holomorphic modular forms**

In 1949, H. Maass introduced these forms to study $L$-functions in real quadratic fields.

*The problem:* Are there modular forms corresponding to these $L$-functions?

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**Classic modular forms**

- Holomorphic ($\Delta = 0$).
- Finite vector spaces.

**Non-holomorphic forms**

- Eigenfunctions of $\Delta$.
- Hilbert space (Spectral theory).
Geometrically:

In the upper half plane $\mathbb{H}$, we consider the hyperbolic distance $d$. When a Fuchsian group $\Gamma$ acts on $\mathbb{H}$, the quotient space $\Gamma \backslash \mathbb{H}$ acquires a Riemannian structure.

The non-holomorphic modular forms are the functions of $\Gamma \backslash \mathbb{H}$. 
Non-compact case

Compact case

Introduction to the Spectral Theory of Automorphic Forms
Henryk Iwaniec
Fourier

Any periodic function can be represented by a series of sines and cosines

\[ f(x) = \sum a_n e^{2\pi inx} \]

\[ \Delta e^{2\pi inx} = -4\pi^2 n^2 e^{2\pi inx}, \quad \Delta = d^2/dx^2 \]

Maass

Any automorphic function can be expanded into eigenfunctions

\[ f(z) = \sum a_j u_j(z) + \left( \text{contribution of the continuous spectrum} \right) \]

\[ \Delta u_j = -\lambda_j u_j, \quad \Delta = \text{hyperbolic Laplacian} \]

\[ u_j = \text{Maass form} \]
**Fourier**

Any periodic function can be represented by a series of sines and cosines

\[ f(x) = \sum a_n e^{2\pi inx} \]

\[ \Delta e^{2\pi inx} = -4\pi^2 n^2 e^{2\pi inx}, \quad \Delta = d^2/dx^2 \]

**Maass**

Any automorphic function can be expanded into eigenfunctions

\[ f(z) = \sum a_j u_j(z), \quad \text{if } \Gamma \backslash \mathbb{H} \text{ is compact.} \]

\[ \Delta u_j = -\lambda_j u_j, \quad \Delta = \text{hyperbolic Laplacian} \]

\[ u_j = \text{Maass form} \]
- **The constant eigenfunction:** $u_0(z) = |\Gamma \backslash \mathbb{H}|^{-1/2}$
- **First nontrivial Maass forms:**

  ![Graph of $u_0(z)$](image1)
  ![Graph of $u_1(z)$](image2)
  ![Graph of $u_2(z)$](image3)
  ![Graph of $u_3(z)$](image4)
Automorphic kernel

Given a function \( k : [0, \infty) \rightarrow \mathbb{R} \)

\[
K(z, w) = \sum_{\gamma \in \Gamma} k(d(\gamma z, w)), \quad z, w \in \mathbb{H}
\]

is automorphic in \( z \) and \( w \): 
\[
K(z, w) = K(\gamma z, w) = K(z, \gamma w).
\]

Pretrace formula

\[
K(z, w) = \sum_{j \geq 0} h(t_j) u_j(z) \overline{u_j(w)} + \ldots
\]

where \( h \) is the Selberg transform of \( k \) (up to a change of variables).
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The Kuznetsov formula \( \text{SL}_2(\mathbb{Z}) \)

\[
\sum_j h(t_j) \nu_j(n) \overline{\nu_j(m)} + \cdots = \sum_{c=1}^{\infty} \frac{1}{c} S(n, m; c) H\left(\frac{4\pi \sqrt{|mn|}}{c}\right) + \cdots
\]

- A consequence of the Kuznetsov formula is that there is cancellation among Kloosterman sums for different moduli.

- This can also be used to deduce spectral results from arithmetic results via Kloosterman sums.
The Kuznetsov formula

\[
\sum_j h(t_j) \nu_j(n) \overline{\nu_j(m)} + \cdots = \sum_{c=1}^{\infty} \frac{1}{c} S(n, m; c) H\left(\frac{4\pi \sqrt{|mn|}}{c}\right) + \cdots
\]

\[
H(x) = \begin{cases} 
2i \int_{-\infty}^{\infty} \text{th}(t) \frac{J_{2it}(x)}{\cosh(\pi t)} \ dt, & \text{if } mn > 0 \\
\frac{4}{\pi} \int_{-\infty}^{\infty} \text{th}(t) K_{2it}(x) \sinh(\pi t) \ dt, & \text{if } mn < 0
\end{cases}
\]

- Asymmetry between the cases \( mn > 0 \) and \( mn < 0 \).
- Difficulties to invert the integral transform \( h \rightarrow H \).
The kernel of these transforms is by no means simple

\[ f(x, t) = e^{\pi t/2} K_{it}(x) \]
The Kuznetsov formula

\[ \sum_j h(t_j) \nu_j(n) \overline{\nu_j(m)} + \cdots = \sum_{c=1}^{\infty} \frac{1}{c} S(n, m; c) H\left( \frac{4\pi \sqrt{|mn|}}{c} \right) + \cdots \]

Theorem

For all \( x > 0 \), \( H(x) = G(x) \) where

\[ G(x) = 4\pi x \int_0^{\infty} k(r) J_0(x\sqrt{r + \epsilon_0}) \, dr, \]

with \( \epsilon_0 = 1 \) if \( mn > 0 \) and \( \epsilon_0 = 0 \) if \( mn < 0 \).
Why a new formulation?

The application of the Kuznetsov formula becomes simpler than with the original statement, because the transforms $h \rightarrow k$ and $k \rightarrow G$ are almost as simple as Fourier transforms.

\[
\begin{align*}
B_0 \text{ bounds } \hat{h} & \quad \Rightarrow \quad k^2 (\sinh^2 \frac{x}{2}) \leq C \frac{B_0(x)B_1(x)}{\sinh x} \\
B_1 \text{ bounds } \hat{h}' &
\end{align*}
\]

Example:

For $h(t) = e^{-t^2/T^2}$ we obtain a quick proof of

\[
\sum |\nu_j(n)|^2 e^{-t_j^2/T^2} + \cdots \sim \pi^{-1} T^2
\]

uniformly for $|n| < CT^{2-\delta}$, $\delta > 0$. 
Why a new formulation?

- The application of the Kuznetsov formula becomes simpler than with the original statement because the transforms $h \rightarrow k$ and $k \rightarrow G$ are almost as simple as Fourier transforms.
- We find an extra-short and natural proof of the Kuznetsov formula. This proof avoids any knowledge about special functions except the definition of $J_0$.
Why a new formulation?

- The application of the Kuznetsov formula becomes simpler than with the original statement because the transforms $h \rightarrow k$ and $k \rightarrow G$ are almost as simple as Fourier transforms.
- We find an extra-short and natural proof of the Kuznetsov formula. This proof avoids any knowledge about special functions except the definition of $J_0$.
- It allows to use pairs $k$ and $h$ given by closed formulas.

Example:

$$G(x) = 4\pi x \mu^{-1} e^{-x^2/4\mu}, \quad \mu > 0$$

$$k(r) = e^{-\mu r} \quad \leftrightarrow \quad h(t) = 4e^{\mu/2} \sqrt{\frac{\pi}{\mu}} K_{it}(\mu/2)$$
**Why a new formulation?**

- The application of the Kuznetsov formula becomes simpler than with the original statement because the transforms $h \rightarrow k$ and $k \rightarrow G$ are almost as simple as Fourier transforms.

- We find an extra-short and natural proof of the Kuznetsov formula. This proof avoids any knowledge about special functions except the definition of $J_0$.

- It allows to use pairs $k$ and $h$ given by closed formulas.

- The reversed Kuznetsov formula becomes more natural. We can think of it as a Fourier inversion.

$$G(x) = 4\pi x \int_0^\infty k(r) J_0(x\sqrt{r + \epsilon_0}) \, dr$$
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Two well-known examples of approximate identities

Both are linked to the Fourier expansion of classical modular forms. The first through the function $\theta$ and the second through the $j$-invariant.

1

$$\frac{1}{4} \left( \sum_{n=-15}^{15} e^{-n^2/4} \right)^2 = 3.141592653589793328 \ldots$$

$$\pi = 3.141592653589793238 \ldots$$

2

$$e^{\pi \sqrt{163}} = 262537412640768743.999999999999999250 \ldots$$

$$744 + 640320^3 = 262537412640768744$$

F. Chamizo and D. Raboso, *Modular forms and almost integers* (Spanish).
Two examples of our identities

\[ r(n) = \#\{(a, b) \in \mathbb{Z}^2 : a^2 + b^2 = n\} \]

\[ S = \sum_{n=0}^{\infty} (3 + (-1)^n) \frac{r(n)r(n+4)}{2(n+4)^2}, \quad J = \int_{-\infty}^{\infty} \frac{\frac{1}{4} + t^2}{\cosh(\pi t)} |f(t)|^2 \, dt \]

\[ f(t) = \zeta(s) L(s, \chi_4) / \zeta(2s) \quad \text{with} \quad s = \frac{1}{2} + it. \]

1.
\[ \frac{S - 3}{J} = \pi - \epsilon \quad \text{with} \quad 0 < \epsilon < 4 \cdot 10^{-14}. \]

2.
\[ \sum_{n=1}^{\infty} r(n)r(3n + 2)\sqrt{n}e^{-\left(\frac{1}{4} \log n\right)^2} = 72e^9 \sqrt{\pi}(1 - \epsilon), \quad \epsilon \approx 3 \cdot 10^{-7}. \]
Spectral methods
Combinatorial methods
Analytical methods

On the Kuznetsov formula
Exotic approximate identities and Maass forms

Spectral theory (pretrace formula)

\[ K(z, w) = \sum_{\gamma \in \Gamma} k(d(\gamma z, w)) = a_0 + a_1 u_1(z) u_1(w) + \cdots \approx a_0. \]

We choose \( k, \Gamma, z \) and \( w \) such that \( K(z, w) \) has an arithmetically meaning.

1. The group is \( \Gamma = \text{SL}_2(\mathbb{Z}) \). The error depends on the third eigenvalue (\( \lambda_3 = 190.13 \)) due to certain symmetries of the eigenfunctions.

2. The group is used to construct Shimura curve \( X(6, 1) \). The error depends on the first eigenvalue (\( \lambda_1 = 6.96 \)) because in this case there are no symmetries.
Where do the products $r(n)$ come from?

$$K(z, w) = \sum_{\gamma \in \Gamma} k(d(\gamma z, w)).$$

It turns out that $d\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} i, i\right)$ is a function of $a^2 + b^2 + c^2 + d^2$ and $ad - bc = 1$. Note that

$$\begin{cases} a^2 + b^2 + c^2 + d^2 = n \\ ad - bc = 1 \end{cases} \iff \begin{cases} (a-d)^2 + (c+b)^2 = n-2 \\ (a+d)^2 + (c-b)^2 = n+2 \end{cases}$$

and the number of solutions is essentially $r(n+2)r(n-2)$.

H. Iwaniec, *Spectral Methods of Automorphic Forms.*
In the second example $r(n)$ appears using a quaternion group:

$$G_3 = \left\{ \frac{1}{2} \begin{pmatrix} a + b\sqrt{3} & c + d\sqrt{3} \\ -c + d\sqrt{3} & a - b\sqrt{3} \end{pmatrix} \in \text{SL}_2(\mathbb{R}) \right\}$$

with $a, b, c, d \in \mathbb{Z}$ of the same parity.

The equations become

$$\begin{cases} 
3(b^2 + d^2) = m \\
 a^2 + c^2 = m + 4 
\end{cases} \quad \xrightarrow{m=6n} \quad \begin{cases} 
b^2 + d^2 = 2n \\
 a^2 + c^2 = 6n + 4 
\end{cases}$$

Using $r(n) = r(2n)$, the number of solutions is $r(n)r(3n + 2)$. 
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Rowland’s Sequence

\[ a_k = a_{k-1} + \gcd(k, a_{k-1}) \quad \text{with} \quad a_1 = 7. \]

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>…</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_k )</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>15</td>
<td>18</td>
<td>19</td>
<td>20</td>
<td>21</td>
<td>22</td>
<td>33</td>
<td>…</td>
</tr>
<tr>
<td>( a_k - a_{k-1} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>11</td>
<td>…</td>
</tr>
</tbody>
</table>

Theorem

E.S. Rowland

\[ a_k - a_{k-1} \text{ is 1 or prime for every } k \geq 1. \]

J. Integer Seq., 11(2): Article 08.2.8, 13, 2008.
Auxiliary sequences

\[
\begin{align*}
\begin{cases}
    c_n^* &= c_{n-1}^* + \text{lfp}(c_{n-1}^*) - 1 \\
    c_1^* &= 5
\end{cases}
\quad \text{and} \quad r_n^* &= \frac{c_n^* + 1}{2}
\end{align*}
\]

where lfp(·) is the least prime factor of an integer.

Proposition

\[
a_k - a_{k-1} = \begin{cases}
    \text{lfp}(c_{n-1}^*), & \text{if } k = r_n^* \text{ for some } n > 1. \\
    1, & \text{otherwise.}
\end{cases}
\]

\[
\{a_k - a_{k-1}\}_{k>1} \text{ contains infinitely many primes.}
\]
### Generalized Rowland’s sequence

The generalized Rowland’s sequence is defined by:

\[ a_k = a_{k-1} + \gcd(k, a_{k-1}) \quad \text{with} \quad a_1 > 3 \text{ odd}. \]

Here is a table showing the first few terms of the sequence:

<table>
<thead>
<tr>
<th>k</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_k)</td>
<td>805</td>
<td>806</td>
<td>807</td>
<td>808</td>
<td>809</td>
<td>810</td>
<td>811</td>
<td>812</td>
<td>813</td>
</tr>
<tr>
<td>(a_k - a_{k-1})</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
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</table>

The table continues with:

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<tr>
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<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>...</th>
</tr>
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<tbody>
<tr>
<td>814</td>
<td>825</td>
<td>828</td>
<td>829</td>
<td>830</td>
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<td>836</td>
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<td>...</td>
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<td>5</td>
<td>1</td>
<td>1</td>
<td>9</td>
<td>...</td>
</tr>
</tbody>
</table>
Auxiliary sequences

\[
\begin{align*}
r_{n+1} &= \min \left\{ k > r_n : \gcd(k, c_n) \neq 1 \right\} \\
r_1 &= 1
\end{align*}
\]
\[
\begin{align*}
c_{n+1} &= c_n + \gcd(c_n, r_{n+1}) - 1 \\
c_1 &= a_1 - 2
\end{align*}
\]

Proposition

\[
a_k = c_n + k + 1 \quad \text{for} \quad r_n \leq k < r_{n+1}.
\]

\[
a_k - a_{k-1} = \begin{cases} 
\gcd(c_{n-1}, r_n), & \text{if} \quad k = r_n \quad \text{for some} \quad n > 1. \\
1, & \text{otherwise.}
\end{cases}
\]
**Conjecture A**

For any generalized Rowland’s sequence, there exists a positive integer $N$ such that $a_k - a_{k-1}$ is 1 or prime for every $k > N$.

Fixed $a_1 > 3$ odd, the Conjecture A holds if any of these conditions is satisfied:

- There is an $n$ such that $2r_n - 1 = c_n$.
- There is an $m$ such that $c_m$ is prime.

$\quad r_{n+1} = \min \{ k > r_n : \gcd(k, c_n) \neq 1 \}$, \quad $c_{n+1} = c_n + \gcd(c_n, r_{n+1}) - 1$
### Rowland’s Sequence

**Distributional properties of powers of matrices**

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
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<th>10</th>
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<tbody>
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<td>11</td>
<td>12</td>
<td>23</td>
<td>24</td>
<td>47</td>
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<td>...</td>
</tr>
<tr>
<td>$c_n$</td>
<td>5</td>
<td>9</td>
<td>11</td>
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<td>45</td>
<td>47</td>
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<td>95</td>
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<table>
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<tr>
<th>$n$</th>
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<tbody>
<tr>
<td>$r_n$</td>
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<td>3</td>
<td>5</td>
<td>6</td>
<td>41</td>
<td>42</td>
<td>83</td>
<td>84</td>
<td>167</td>
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</tr>
<tr>
<td>$c_n$</td>
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<td>41</td>
<td>81</td>
<td>83</td>
<td>165</td>
<td>167</td>
<td>333</td>
<td>335</td>
<td>...</td>
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<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
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<th>7</th>
<th>8</th>
<th>9</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_n$</td>
<td>1</td>
<td>5</td>
<td>7</td>
<td>10</td>
<td>12</td>
<td>131</td>
<td>132</td>
<td>263</td>
<td>264</td>
<td>...</td>
</tr>
<tr>
<td>$c_n$</td>
<td>115</td>
<td>119</td>
<td>125</td>
<td>129</td>
<td>131</td>
<td>261</td>
<td>263</td>
<td>525</td>
<td>527</td>
<td>...</td>
</tr>
</tbody>
</table>

$\left< \diamond \ \diamond \right>$

$r_{n+1} = \min \{ k > r_n : \gcd(k, c_n) \neq 1 \}, \quad c_{n+1} = c_n + \gcd(c_n, r_{n+1}) - 1$
**Conjecture A**

For any generalized Rowland’s sequence, there exists a positive integer $N$ such that $a_k - a_{k-1}$ is 1 or prime for every $k > N$.

\[
n_0 = \inf\{n \in \mathbb{Z}^+ : c_n = 2r_n - 1\}, \quad m_0 = \inf\{n \in \mathbb{Z}^+ : c_n \text{ is prime}\}.
\]

**Conjecture B**

(i) $n_0 < \infty$,  
(ii) $m_0 < \infty$,  
(iii) $n_0 = m_0 + 1 < \infty$.

Conjecture B $\Rightarrow$ Conjecture A

\[
r_{n+1} = \min\{k > r_n : \gcd(k, c_n) \neq 1\}, \quad c_{n+1} = c_n + \gcd(c_n, r_{n+1}) - 1
\]
Rowland’s chains

They are finite sublists of primes inside of a sequence \( \{a_k - a_{k-1}\} \).

For \( a_1 = 7 \), the first 15 primes of the sequence are

\[
C_{15} = \{5, 3, 11, 3, 23, 3, 47, 3, 5, 3, 101, 3, 7, 11, 3\}.
\]

We give a characterization which allows to verify whether \( C_m \) is a Rowland’s chain. For example:

- \( C_4 = \{3, 19, 5, 3\} \) (✓)
- \( C_3 = \{17, 5, p\} \quad \forall \ p > 3 \) (✗)
- \( C_{2m} = \{p_1, \ldots, p_m, p_1, \ldots, p_m\} \) with \( p_1, \ldots, p_m \) distinct primes. (✗)
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We start with an example

- We choose a “large” prime

\[ p = 2311. \]

- Given a matrix \( M \), we take the pseudorandom points

\[
\begin{pmatrix}
  x_n \\
  y_n
\end{pmatrix} = M^n \begin{pmatrix}
  x_0 \\
  y_0
\end{pmatrix}
\]

reduced modulo \( p \).

\[ \exp_p(M) = \text{order of } M \text{ modulo } p. \]
\[ p = 2311 \]

\[ A = \begin{pmatrix} 703 & 633 \\ 934 & 841 \end{pmatrix} \quad B = \begin{pmatrix} 704 & 635 \\ 653 & 589 \end{pmatrix} \quad C = \begin{pmatrix} 703 & 787 \\ 862 & 965 \end{pmatrix} \]

\[ \exp_p(A) = p - 1 \quad \exp_p(B) = \frac{p - 1}{2} \quad \exp_p(C) = \frac{p - 1}{154} \]
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Rowland’s Sequence
Distributional properties of powers of matrices

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\[\exp_p(A) = p - 1\]
\[\exp_p(B) = \frac{p - 1}{2}\]
\[\exp_p(C) = \frac{p - 1}{154}\]
By changing the prime

\[ C = \begin{pmatrix} 703 & 787 \\ 862 & 965 \end{pmatrix} \]

- \( p = 2333 \) 
  \[ \exp_p(C) = \frac{p + 1}{2} \]

- \( p = 2309 \) 
  \[ \exp_p(C) = \frac{p - 1}{4} \]

- \( p = 2311 \) 
  \[ \exp_p(C) = \frac{p - 1}{154} \]
Types of sieving

Sieve of Eratosthenes
- classical sieve

The large sieve

The larger sieve

Primes

2, 3, 4, 5, 6, 7, 8, 9, 10,
11, 12, 13, 14, 15, 16, 17, 18, 19, 20,
21, 22, 23, 24, 25, 26, 27, 28, 29, 30,
31, 32, 33, 34, 35, 36, 37, 38, 39, 40,
41, 42, 43, 44, 45, 46, 47, 48, 49, 50,
51, 52, 53, 54, 55, 56, 57, 58, 59, 60,
61, 62, 63, 64, 65, 66, 67, 68, 69, 70,
71, 72, 73, 74, 75, 76, 77, 78, 79, 80,
81, 82, 83, 84, 85, 86, 87, 88, 89, 90,
...
Types of sieving

Sieve of Eratosthenes
classical sieve

The large sieve

The larger sieve

Primes

\[
2, 3, \not{2}, 5, \not{3}, 7, \not{5}, 9, 10, 11, \not{2}, 13, \not{3}, 15, \not{5}, 17, \not{3}, 19, 20, 21, \not{2}, 23, \not{5}, 25, \not{3}, 27, \not{5}, 29, \not{3}, 30, 31, \not{3}, 33, \not{5}, 35, \not{3}, 37, \not{5}, 39, \not{3}, 40, 41, \not{2}, 43, \not{5}, 45, \not{3}, 47, \not{5}, 49, \not{3}, 50, 51, \not{3}, 53, \not{5}, 55, \not{3}, 57, \not{5}, 59, \not{3}, 60, 61, \not{2}, 63, \not{3}, 65, \not{5}, 67, \not{3}, 69, \not{5}, 70, 71, \not{2}, 73, \not{3}, 75, \not{5}, 77, \not{3}, 79, \not{5}, 80, 81, \not{2}, 83, \not{3}, 85, \not{5}, 87, \not{3}, 89, \not{5}, 90, \ldots
\]

\[2 \mid n\]
Types of sieving

Sieve of Eratosthenes
classical sieve

The large sieve

The larger sieve

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>2, 3, 5, 7, 9,</td>
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<td>71, 73, 75, 77, 79,</td>
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<tr>
<td>81, 83, 85, 87, 89,</td>
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<td>...</td>
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</tbody>
</table>

3 | n
Types of sieving

Sieve of Eratosthenes
- classical sieve

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<tr>
<td>2, 3,</td>
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The large sieve

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<tr>
<td>5, 7,</td>
</tr>
<tr>
<td>37, 47,</td>
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The larger sieve

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We eliminate one class by prime.
Types of sieving

Sieve of Eratosthenes
- classical sieve

The large sieve
- The larger sieve

Squares

2, 3, 4, 5, 6, 7, 8, 9, 10,
11, 12, 13, 14, 15, 16, 17, 18, 19, 20,
21, 22, 23, 24, 25, 26, 27, 28, 29, 30,
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...

Dulcinea Raboso
Spectral, combinatorial and analytic methods in number theory
Types of sieving

Sieve of Eratosthenes
- classical sieve

The large sieve
- The larger sieve

Squares

\[ \mathcal{P}, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, \ldots \]

\[ \left( \frac{n}{3} \right) = -1 \]
Types of sieving

Sieve of Eratosthenes
- classical sieve

The large sieve
- The larger sieve

Squares

\[ \sqrt{1}, 4, 6, \sqrt{7}, 9, 10, \]
\[ 12, 13, 15, 16, 18, 19, 21, 22, 24, 25, 27, 28, 30, 21, 22, 24, 25, 27, 28, 30, 31, 33, 34, 36, 37, 39, 40, 42, 43, 45, 46, 48, 49, 51, 52, 55, 57, 58, 60, 61, 63, 64, 66, 67, 69, 70, 72, 73, 75, 76, 78, 79, 80, 81, 82, 84, 85, 87, 88, 90, \]
\[ \cdots \]

\[ \left( \frac{n}{5} \right) = -1 \]
Types of sieving

Sieve of Eratosthenes
   classical sieve

The large sieve

The larger sieve

Squares

4, 16, 25, 36, 49, 64, ...

We eliminate many classes by prime.
Types of sieving

Sieve of Eratosthenes
  classical sieve

The large sieve

We eliminate many more classes by prime.
Types of sieving

Sieve of Eratosthenes
classical sieve

The large sieve

The larger sieve

Number of classes close to prime.

\[
\exp_p(n) = \begin{cases} 
\text{order of } n \text{ in } \mathbb{F}_p^* \\ 
0 & \text{if } p \mid n 
\end{cases}
\]

It is very unlikely to find \( n \) such that \( \exp_p(n) \) is small for many consecutive primes.

For \( p \) in a reasonably large range, \( k \rightarrow n^k \pmod{p} \) is a good pseudorandom number generator for almost any choice of \( n \).

Set (Interval):

\[ \text{GL}_2(\mathbb{Z})[N] = \{ A \in \text{GL}_2(\mathbb{Z}) : 0 \leq a_{ij} \leq N \} \]

Choose \(0 < \theta < \gamma\)

Primes:

\[ \{ p \text{ prime} : p < N^\gamma \} \]

Elements that remain after sieving:

\[ \{ A \in \text{GL}_2(\mathbb{Z})[N] : \exp_p(A) \leq N^\theta, \ p < N^\gamma \} \]
For a fixed prime $p$, we consider

$$\{ A \in \text{GL}_2(\mathbb{F}_p) : \det A = m \}$$

How many matrices have $\exp_p(A) = n$?

- **Diagonalizable case**

$$\begin{pmatrix} \alpha & 0 \\ 0 & m\alpha^{-1} \end{pmatrix} \quad \alpha \in \mathbb{F}_p^*$$

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^p \end{pmatrix} \quad \alpha \in \mathbb{F}_{p^2} - \mathbb{F}_p$$

- **Non-diagonalizable case**

$$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \quad \alpha \in \mathbb{F}_p^*$$

**Canonical form** $\longleftrightarrow$ **Trace**
Gallagher’s sieve applied to the traces gives

**Theorem**

Given $\varepsilon > 0$ and $0 < \theta < \gamma \leq 1$, the number of matrices $A \in GL_2(\mathbb{Z})[N]$ such that $\exp_{p}(A) \leq N^\theta$ for all $p < N^\gamma$, is

$$< CN^{2\theta+1+\varepsilon}$$

**Theorem**

Under the same conditions, with $A \in SL_2(\mathbb{Z})[N]$, the number of matrices is

$$< CN^{\theta+1+\varepsilon}$$

Using exponential sum techniques we also prove that there are “nearby” matrices with the same order.
I. **Spectral methods**
- On the Kuznetsov formula
- Exotic approximate identities and Maass forms

II. **Combinatorial methods**
- Rowland’s Sequence
- Distributional properties of powers of matrices

III. **Analytical methods**
- Van der Corput’s method and optical illusions
- Lattice points in the 3-dimensional torus
Dulcinea Raboso  
Spectral, combinatorial and analytic methods in number theory

I. Spectral methods
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Many problems in number theory lead to the estimation of trigonometric sums

\[ S = \sum_{a \leq n \leq b} e(f(n)), \]

where \( e(x) = e^{2\pi i x} \) and \( f \) is a real function.

**Van der Corput’s method**

- **A-process:** This corresponds to divide the range of summation applying Cauchy’s inequality to reduce the oscillations, at the cost of certain loss of accuracy in the estimation.
- **B-process:** We transform the new sum by Poisson summation combined with the stationary phase principle.
Some examples

1. If $f'' \asymp \lambda$,
   \[ \sum_{n=1}^{N} e(f(n)) < C(N\lambda^{1/2} + \lambda^{-1/2}). \]

2. If $f'$ is monotonic and $|f'| \leq 1/2$,
   \[ \sum_{n=1}^{N} e(f(n)) = \int_{1}^{N} e(f(x)) \, dx + O(1). \]
Our trigonometric sum

\[ S(N; \alpha) = \sum_{n=1}^{N} e(\alpha \sqrt{n}) \quad \text{with } \alpha > 0 \text{ fixed.} \]

The derivative of \( \alpha \sqrt{x} \) decreases to 0, so we expect a good approximation by

\[
\int_{1}^{N} e(\alpha \sqrt{x}) \, dx = \frac{\sqrt{N}}{\pi \alpha} e(\alpha \sqrt{N} - 1/4) + O(1).
\]
Consequently a spiral should show up for every $\alpha$,

$$\left\{ S(n; \alpha) \right\}_{n=1}^{N} \quad \text{with} \quad t \in [1, 2\pi \alpha \sqrt{N}]$$

$$\frac{1}{2}(\pi \alpha)^{-2} t(\sin t, -\cos t)$$
**But...**

\[ \alpha = 1 \]

\[ \alpha = \frac{65}{64} \]
But... 

Wait...
The approximation of the exponential sum

\[ S(x; \alpha) = A(x; \alpha) + \text{(translation)}, \]

where

\[ A(x; \alpha) = \frac{e^{\left(\alpha \sqrt{x} - 1/4\right)}}{\pi \alpha} \left(\sqrt{x} + i \cosh \log(\pi \alpha)\right), \]

that when \( x \) varies approximates an Archimedean spiral of width \( 1/\pi \alpha \).

**Optical illusions?**

- The separation between successive turns tends to be \( 1/\pi \alpha^2 \).
- When \( \alpha > \pi^{-1} \) the width of the spiral is smaller than the distance between consecutive values of the discretization.
- \( A(n_1; \alpha) \) and \( A(n_2; \alpha) \) with \( n_1, n_2 \in \mathbb{Z}^+ \) become geometrically consecutive if \( \alpha \sqrt{n_1} \approx \alpha \sqrt{n_2} + 1 \).
**Branch**

It is a sequence \( \{ A(t_k; \alpha) \}_{k=0}^{\infty} \) where \( t_k \) satisfies the recurrence relation

\[
t_{k+1} = t_k + \left[ \frac{2\alpha \sqrt{t_k} + 1}{\alpha^2} + \frac{1}{2} \right].
\]

\( \alpha = 1 \quad \alpha = \sqrt{3} \quad \alpha = 1.3 \)
The recurrence relation \((\alpha^2 = n)\)

\[ t_{k+1} = t_k + \left[ \frac{2\sqrt{nt_k} + 1}{n} + \frac{1}{2} \right]. \]

We find an explicit solution of the recurrence when \(n\) is even.

The simplest case is \(n = 2\),

\[ t_k = \frac{k(k + 1)}{2} + \left\lfloor \sqrt{2t_0} \right\rfloor k + t_0. \]

What happens if \(n\) is odd?
Van der Corput’s method and optical illusions
Lattice points in the 3-dimensional torus

Spectral methods
Combinatorial methods
Analytical methods

$\alpha = 1$
$\alpha = \sqrt{3}$
$\alpha = \sqrt{5}$

$\alpha = \sqrt{2}$
$\alpha = \sqrt{6}$
$\alpha = \sqrt{10}$
Van der Corput’s method and optical illusions
Lattice points in the 3-dimensional torus

$\alpha = 1$

$\alpha = \sqrt{3}$

$\alpha = \sqrt{5}$

$\alpha = \sqrt{2}$

$\alpha = \sqrt{6}$

$\alpha = \sqrt{10}$
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\[ \alpha = 1 \]
\[ \alpha = \sqrt{3} \]
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Spectral methods
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Analytical methods

Van der Corput’s method and optical illusions
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\[
\alpha = 1 \quad \alpha = \sqrt{3} \quad \alpha = \sqrt{5}
\]

\[
\alpha = \sqrt{2} \quad \alpha = \sqrt{6} \quad \alpha = \sqrt{10}
\]
Spectral methods
Combinatorial methods
Analytical methods

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\[ \alpha = 1 \]
\[ \alpha = \sqrt{3} \]
\[ \alpha = \sqrt{5} \]
\[ \alpha = \sqrt{2} \]
\[ \alpha = \sqrt{6} \]
\[ \alpha = \sqrt{10} \]
Van der Corput’s method and optical illusions
Lattice points in the 3-dimensional torus

\[
\begin{align*}
\alpha &= 1 \\
\alpha &= \sqrt{2} \\
\alpha &= \sqrt{5} \\
\alpha &= \sqrt{6} \\
\alpha &= \sqrt{10}
\end{align*}
\]
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Lattice points

The problem: Estimation of the number of points with integer coordinates in large closed domains.

For example, given a domain $\mathcal{D} \in \mathbb{R}^2$, we study the number of points of $\mathbb{Z}^2$ in $R\mathcal{D}$ when $R \in \mathbb{R}^+$ increases.

The number of lattice points in $R\mathcal{D}$ is

$$\sum_{\vec{n} \in \mathbb{Z}^2} \chi(R^{-1}\vec{n}) = R^2 \sum_{\vec{n} \in \mathbb{Z}^2} \hat{\chi}(R\vec{n}),$$

where $\chi$ is the characteristic function of $\mathcal{D}$.

$\vec{n} = \vec{0}$ ←-----→ main term: $|\mathcal{D}|R^2$.
$\vec{n} \neq \vec{0}$ ←-----→ error term.
The circle problem \footnote{M.N. Huxley}

\[
\#\{\vec{n} \in \mathbb{Z}^2 : \|\vec{n}\| \leq R\} = \pi R^2 + O_\epsilon\left(R^{131/208+\epsilon}\right)
\]

for every \(\epsilon > 0\).

The sphere problem \footnote{D.R. Heath-Brown}

\[
\#\{\vec{n} \in \mathbb{Z}^3 : \|\vec{n}\| \leq R\} = \frac{4}{3}\pi R^3 + O_\epsilon\left(R^{21/16+\epsilon}\right)
\]

for every \(\epsilon > 0\).
Lattice points in the \( R \)-scaled torus

\[ \mathbb{T} = \left\{ (x, y, z) \in \mathbb{R}^3 : (\rho' - \sqrt{x^2 + y^2})^2 + z^2 \leq \rho^2 \right\} \]

where \( 0 < \rho < \rho' \) are fixed constants. Say \( \rho' = 1 \).

\[ \mathcal{N}(R) = \#\{ \vec{n} \in \mathbb{Z}^3 : R^{-1} \vec{n} \in \mathbb{T} \}, \quad R > 1 \]

**Theorem**

\[ \mathcal{N}(R) = |\mathbb{T}| R^3 + M_R R^{3/2} + O_\varepsilon (R^{4/3 + \varepsilon}) \]

for every \( \varepsilon > 0 \), where \( M_R \) is a bounded periodic function.
Poisson summation formula

\[ \mathcal{N}(R) = \sum_{\vec{n} \in \mathbb{Z}^3} \chi(R^{-1} \vec{n}) \quad \text{“=”} \quad R^3 \sum_{\vec{n} \in \mathbb{Z}^3} \hat{\chi}(R \vec{n}). \]

\[ \vec{n} = (0, 0, 0) \quad \text{←-----→} \quad \text{main term} \quad \hat{\chi}(0) R^3 \]

\[ \vec{n} = (0, 0, n) \quad \text{←-----→} \quad \text{secondary main term} \quad R^3 \sum_{n \neq 0} \hat{\chi}(0, 0, Rn) \]

Otherwise \quad \text{←-----→} \quad \text{error term} \quad R^3 \sum_{n=\infty}^{\infty} \sum_{m=1}^{\infty} r(m) \hat{\chi}(0, R \sqrt{m}, Rn) \]
**Poisson summation formula**

\[ \mathcal{N}(R) = \sum_{\vec{n} \in \mathbb{Z}^3} \chi(R^{-1} \vec{n}) \quad \approx \quad R^3 \sum_{\vec{n} \in \mathbb{Z}^3} \hat{\chi}(R \vec{n}) \]

<table>
<thead>
<tr>
<th>( \vec{n} )</th>
<th>Main term</th>
<th>Secondary main term</th>
<th>Error term</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (0, 0, 0) )</td>
<td>[ 2\pi^2 \rho^2 R^3 ]</td>
<td>[ 4\pi \rho R^2 \sum_{n=1}^{\infty} \frac{J_1(2\pi R \rho n)}{n} ]</td>
<td>[ O\left(R^{4/3+\epsilon}\right) ]</td>
</tr>
<tr>
<td>( (0, 0, n) )</td>
<td></td>
<td>[ 4\pi R^2 \sum_{n=1}^{\infty} \frac{J_1(2\pi R \rho n)}{n} ]</td>
<td></td>
</tr>
<tr>
<td>Otherwise</td>
<td></td>
<td></td>
<td>[ O\left(R^{4/3+\epsilon}\right) ]</td>
</tr>
</tbody>
</table>
An idea about the estimation of the error term

- Stationary phase principle to get a new exponential sum.
- Using the symmetries we “glue” the variables.
- After some manipulations the sum becomes one appearing in the sphere problem.

Geometrically

The sections of a torus and a sphere are alike and differ in a translation which introduce a phase in the Fourier transform side and is eliminated with Cauchy’s inequality.

References

Part I


Part II


Part III