

*Answers for the Problems on “**Superfluidity in Ultracold Fermi Gases**”*  
(1 tutorial class at “Physics by the Lake”)

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## 1 The ideal Fermi gas

In the LDA, the Fermi distribution function is given by

$$f_F(\mathbf{r}, \mathbf{k}) = \frac{1}{\exp[\beta(\epsilon_{\mathbf{k}} + V(\mathbf{r}) - \mu)] + 1}, \quad (1)$$

where  $\epsilon_{\mathbf{k}} = \mathbf{k}^2/2m$ ,  $V(\mathbf{r})$  is the harmonic trapping potential. As the chemical potential  $\mu$  is fixed by the number of particle equation:

$$N = \int d\mathbf{r} n(\mathbf{r}) = \int d\mathbf{r} \int \frac{d\mathbf{k}}{(2\pi)^3} f_F(\mathbf{r}, \mathbf{k}), \quad (2)$$

one can obtain the expression of the Fermi energy from  $\varepsilon_F = \mu(T = 0)$  and from

$$\int_0^{\varepsilon_F} d\epsilon \sqrt{\epsilon} (\varepsilon_F - \epsilon)^{3/2} = \frac{\pi}{16} \varepsilon_F^3.$$

## 2 BCS theory

Remember the properties of the fermionic operators

$$\{c_{\mathbf{k}\sigma}, c_{\mathbf{k}'\sigma'}^\dagger\} = \delta_{\mathbf{k},\mathbf{k}'} \delta_{\sigma,\sigma'} \quad \{c_{\mathbf{k}\sigma}^\dagger, c_{\mathbf{k}'\sigma'}^\dagger\} = 0 \quad \{c_{\mathbf{k}\sigma}, c_{\mathbf{k}'\sigma'}\} = 0. \quad (3)$$

## 3 BEC-BCS crossover

(a) If we introduce the composite operator

$$b_{\mathbf{q}}^\dagger = \sum_{\mathbf{k}} \varphi_{\mathbf{k}} c_{\mathbf{k}+\mathbf{q}/2\uparrow}^\dagger c_{-\mathbf{k}+\mathbf{q}/2\downarrow},$$

then, defining  $\varphi_{\mathbf{k}} = \tan \theta_{\mathbf{k}}$ , we can rewrite the state  $|\psi\rangle$  as

$$|\psi\rangle \propto \prod_{\mathbf{k}} e^{\varphi_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger} |0\rangle = e^{b_{\mathbf{q}=0}^\dagger} |0\rangle.$$

This state is a coherent state describing the condensation of the composite bosonic particles  $b_{\mathbf{q}}^\dagger$  in the ground state  $\mathbf{q} = 0$ . Note that this is a good description for BEC only at zero temperature, while at finite temperature, the inclusion of thermal fluctuations for pairs (i.e., finite values of  $\mathbf{q}$ ) is essential (see Sec. 2.3.3).

(b) It is useful to introduce the DoS per unit volume at the Fermi surface (see Eqs. (1.11) and (1.17))

$$\mathcal{N}(\varepsilon_F) = \frac{1}{V} N(\varepsilon_F) = \frac{m^{3/2} \sqrt{\varepsilon_F}}{\sqrt{2\pi^2}} = \frac{3}{4} \frac{n}{\varepsilon_F}, \quad (4)$$

in terms of which the scattering length reads  $m/4\pi a = (\pi/2)\mathcal{N}(\varepsilon_F)/k_F a$ .<sup>1</sup> Introducing the dimensionless units of energy,  $x = \epsilon/|\mu|$ ,<sup>2</sup> one can write the gap equation in the form:

$$\frac{\pi}{2k_F a} \mathcal{N}(\varepsilon_F) = \mathcal{N}(|\mu|) \int_0^\infty dx \sqrt{x} \left[ \frac{1}{2x} - \frac{1}{2\sqrt{(x \mp 1)^2 + (\Delta/|\mu|)^2}} \right],$$

where the sign  $\mp$  corresponds respectively to the cases  $\mu > < 0$ .

In the BCS limit, we know from the number equation (2.14) that, as  $\Delta \ll \varepsilon_F$  (weak coupling regime), the chemical potential  $\mu$  differs from the Fermi energy  $\varepsilon_F$  by an amount  $O(\Delta^2/\varepsilon_F^2)$ ,  $\mu \simeq \varepsilon_F$ . Therefore the gap equation now reads

$$\frac{\pi}{2k_F a} = \underbrace{\int_0^\infty dx \sqrt{x} \left[ \frac{1}{2x} - \frac{1}{2\sqrt{(x-1)^2 + (\Delta/\varepsilon_F)^2}} \right]}_{\ln(\varepsilon^2 \Delta / 8\varepsilon_F)},$$

from which one gets the expression (2.19).

In the BEC limit instead  $\mu < 0$  and we have seen that  $\theta_{\mathbf{k}} \simeq \Delta/2\xi_{\mathbf{k}} \ll 1$  (density of particles in the state  $\mathbf{k}$ ), therefore in the first approximation we have that:

$$\frac{\pi}{2k_F a} \mathcal{N}(\varepsilon_F) \simeq \mathcal{N}(|\mu|) \underbrace{\int_0^\infty dx \sqrt{x} \left[ \frac{1}{2x} - \frac{1}{2(x+1)} \right]}_{\pi/2},$$

from which we get that  $\sqrt{|\mu|/\varepsilon_F} = 1/k_F a$  and therefore Eq. (2.15). One can double-check that the energy scales are arranged as  $\varepsilon_F \ll \Delta \ll |\mu|$  by solving the number equation in this limit

$$n \simeq \frac{\mathcal{N}(|\mu|)\Delta^2}{2|\mu|} \underbrace{\int_0^\infty dx \frac{\sqrt{x}}{(x+1)^2}}_{\pi/2},$$

which gives  $\Delta \simeq \sqrt{16/3\pi}\varepsilon_F \sqrt{1/k_F a}$ .<sup>3</sup>

## 4 Magnetised superconductor

Starting from the expression (3.6), consider the gap equation,  $\partial f/\partial \Delta = 0$  and then use the BCS result (1.31) for  $\Delta_{\text{BCS}}$ .

<sup>1</sup> Note that for a single component Fermi gas  $\mathcal{N}(\varepsilon_F) = 3n/2\varepsilon_F$ .

<sup>2</sup> One has:

$$\frac{1}{V} \sum_{\mathbf{k}} = \int \frac{d\mathbf{k}}{(2\pi)^3} = \int_0^\infty d\epsilon \mathcal{N}(\epsilon).$$

<sup>3</sup> One can show that expanding to the next order

$$\left( \frac{\Delta}{\varepsilon_F} \right)^2 \simeq \frac{16}{3\pi} \frac{1}{k_F a} + \frac{4}{3\pi^2} (k_F a)^2.$$