Course in 3 lectures on
Superfluidity in Ultracold Fermi Gases

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During lecture I. ...

1) Fully polarised gas = ideal Fermi gas
   - Emergence of quantum degeneracy
   - Distribution of the gas
   - Energy of the gas
   - Fermi pressure

2) Two-component mixtures can interact!
   - Today: how & how tuning is possible

3) Pairing instability & BCS theory
Why tuning the interaction is interesting?

binding fermionic pairs into molecules

BEC

BCS
II. Feshbach resonances & BEC-BCS crossover
Interactions between atoms

Dilute gases: two-body collisions

particle separation $\gg$ scattering length

$\ n^{-1/3} \sim 100\text{nm}$ \quad \quad $a \sim 5\text{nm}$

1) Essential to ensure thermalization (equilibrium & cooling)
2) Can be tuned!

\[
\left[-\frac{-\nabla^2}{2m_r} + U(r)\right] \psi_k(r) = \frac{k^2}{2m_r} \psi_k(r)
\]

\[
\lim_{r \to \infty, k \to 0} \psi_k(r) \propto 1 - \frac{a}{r} \quad \text{(low-energy asymptotics)}
\]

\[
a \approx \frac{2m_r}{4\pi} \int dr U(r)
\]

\[
\begin{cases}
a<0 \quad \text{attractive effective interaction} \\
a>0 \quad \text{repulsive}
\end{cases}
\]
E.g., square well potential

\[ \psi_k(r) = \frac{u_k(r)}{r} \]

\[ u_0(r) = \begin{cases} c_1(r - a) & r > R_e \\ c_2 \sin(k_0r) & r < R_e \end{cases} \]
In general

1) The sign of $a$ depends on the energy of the highest bound state

2) If there are no bound states $a<0$ (attractive interaction)
In general

1) The sign of \( a \) depends on the energy of the highest bound state.

2) If there are no bound states \( a < 0 \) (attractive interaction).

3) Every time a bound state is formed:

Mathematical expression:

\[ \epsilon_b = -\frac{1}{2m_r a^2} \]
But a scattering potential cannot be changed externally...
Interactions between atoms

- At short distances, the electronic spin of the two atoms can flip and change the initial hyperfine states.

\[
\hat{H}_{\text{int}} = \frac{U_s + 3U_t}{4} + (U_t - U_s) \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2
\]
Feshbach resonances

\[ a(B) = a_{bg} \left( 1 - \frac{\Delta B}{B - B_0} \right) \]
Feshbach resonances

[C. A. Regal et al., Nature 424, 47 (2003)]
Feshbach resonances

Feshbach resonances

[C. A. Regal et al., Nature 424, 47 (2003)]
At $T=0$, described by the same ground state

- weakly repulsive diatomic molecules
- weakly attractive fermionic atoms
T=0 mean-field theory

\[ \hat{\mathcal{H}} = \sum_{\mathbf{k}, \sigma = \uparrow, \downarrow} \varepsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}\sigma} + \frac{1}{V} \sum_{\mathbf{k}, \mathbf{k}', \mathbf{q}} U_{\mathbf{k}, \mathbf{k}'} c_{\mathbf{k}+\mathbf{q}/2}^{\dagger} c_{\mathbf{k}+\mathbf{q}/2}^{\dagger} c_{-\mathbf{k}'+\mathbf{q}/2} c_{-\mathbf{k}'+\mathbf{q}/2} c_{\mathbf{k}'+\mathbf{q}/2}^{\dagger} c_{\mathbf{k}'+\mathbf{q}/2}^{\dagger} \]

- Fix the total number of atoms (grand-canonical): N.B. \( \mu \neq \varepsilon_F \)
  \[ \hat{N} = \sum_{\mathbf{k}, \sigma = \uparrow, \downarrow} c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}\sigma} \]
  \[ \hat{\mathcal{H}} - \mu \hat{N} \]

- The BCS ground state can also describe the BEC limit
  \[ |\psi\rangle = \prod_{\mathbf{k}} \left( \cos \theta_{\mathbf{k}} + \sin \theta_{\mathbf{k}} c_{\mathbf{k}\uparrow}^{\dagger} c_{-\mathbf{k}\downarrow}^{\dagger} \right) |0\rangle \]

- Order parameter \( \Delta_{\mathbf{k}} = -\frac{1}{V} \sum_{\mathbf{k}'} U_{\mathbf{k}, \mathbf{k}'} \langle \psi | c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} |\psi\rangle \)

- Gap & number equation (now solve simultaneously as \( \mu \neq \varepsilon_F \))
  \[ \Delta_{\mathbf{k}} = -\frac{1}{V} \sum_{\mathbf{k}'} U_{\mathbf{k}, \mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{2E_{\mathbf{k}'}} \quad n = \frac{1}{V} \sum_{\mathbf{k}} \left( 1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}} \right) \]
We remember that in the BCS limit

\[ n_k = \sin^2 \theta_k \]

...now however

[J.R. Engelbrecht et al., PRB 55, 15153 (1997).]
T=0 mean-field theory

- Contact interaction

\[ \hat{H} = \sum_{\mathbf{k}, \sigma = \uparrow, \downarrow} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \frac{g}{V} \sum_{\mathbf{k}, \mathbf{k}', \mathbf{q}} c_{\mathbf{k}+\mathbf{q}/2\uparrow}^\dagger c_{-\mathbf{k}+\mathbf{q}/2\downarrow} c_{-\mathbf{k}'} + \mathbf{q}/2\downarrow c_{\mathbf{k}'} + \mathbf{q}/2\uparrow \]

- Introduce the scattering length (T-matrix: see App. B)

\[ m \frac{4\pi a}{4\pi a} = \frac{1}{g} + \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{k_0 = 1/R_e} \frac{1}{2\epsilon_{\mathbf{k}}} \]

- Now no (log) divergence in the gap equation

\[ m \frac{4\pi a}{4\pi a} = \frac{1}{V} \sum_{\mathbf{k}} \left( \frac{1}{2\epsilon_{\mathbf{k}}} - \frac{1}{2E_{\mathbf{k}}} \right) \]

- Solved simultaneously with the number equation

\[ n = \frac{1}{V} \sum_{\mathbf{k}} \left( 1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}} \right) \]
Problem 3

\[ \frac{1}{k_F a} \to -\infty \]

\[ \varepsilon_F \sim \mu \]

\[ \Delta \ll \varepsilon_F \]

\[ \Delta \sim \varepsilon_F \exp \left( -\frac{\pi}{2|a|k_F} \right) \]

from number equation

from gap equation
BEC limit

Problem 3

\[
\frac{1}{k_F a} \rightarrow +\infty
\]

\[
\varepsilon_F \ll \Delta \ll |\mu|
\]

\[
\Delta \simeq \sqrt{\frac{16}{3\pi}} \varepsilon_F \sqrt{\frac{1}{k_F a}}
\]

\[
\mu \simeq \frac{\epsilon_b}{2} = -\frac{1}{2ma^2}
\]

from number equation

from gap equation
Crossover
Spectrum of excitations

\[ E_{\text{gap}} = \min_k \sqrt{(\varepsilon_k - \mu)^2 + \Delta^2} = \begin{cases} \Delta \\ \sqrt{\Delta^2 + \mu^2} \end{cases} \quad \mu > 0 \]
\[ \mu < 0 \]

\[ \sqrt{\Delta^2 + \mu^2} \]

\[ k = \sqrt{2m\mu} \]
Finite T


BEC: condensate forms out of preformed molecules

BCS: pairing instability

\[ \Delta_{BCS} \sim k_B T_{BCS} \ll \varepsilon_F \]

\[ T_{BEC} \sim T_F \ll T_{diss} \]
Finite T


BEC: condensate forms out of preformed molecules

BCS: pairing instability

$\Delta_{\text{BCS}} \sim k_B T_{\text{BCS}} \ll \varepsilon_F$

$T_{\text{BEC}} \sim T_F \ll T_{\text{diss}}$
What is measured in experiments?
Molecule formation

A. Very fast sweep: nothing happens
B. Slow sweep: creation of molecules
C. Very slow sweep: creation of a BEC of molecules

[C. A. Regal et al., Nature 424, 47 (2003)]
BEC of diatomic molecules

- Bimodal distribution for the molecular cloud

\[ T \sim T_c \quad T < T_c \]

[Image of bimodal distribution graphs]

[M. Greiner et al., Nature 426, 537 (2004)]
Condensation on the BCS side

- You would not see much from the density distribution!!

![Diagram showing energy vs. magnetic field with molecular and atomic states represented](attachment:image.png)
Condensation on the BCS side

- BCS of Fermi pairs: probe the condensate by pair-wise projection into molecules

![Graphical representation of energy and magnetic field with molecular and atomic states, illustrating closer to the resonance.

[C. A. Regal et al., PRL 92, 040403 (2004)]
BEC-BCS Crossover in Experiments

[Q.J. Chen et al., PRA 73, 041601 (2006)]
Superfluidity across the resonance

[M.W. Zwierlein et al., Nature 435, 1047 (200)]
Feshbach resonances

[S. Inouye et al., Nature 392, 151 (1998)]

[J. L. Roberts et al., PRL 81, 5109 (1998)]

[S. L. Cornish et al., PRL 85, 1795 (2000)]