Problem set 1

January 30, 2013

to submit by Monday the 11th of February

1 Ideal Bose and Fermi gases

Starting from the expression of the partition function

\[ Z = \text{Tr}[e^{-\beta(H-H\mu N)}] = \prod_{i=0}^{\infty} \sum_{N_i} e^{-\beta(\epsilon_i-\mu)N_i}, \]  

(1)

in the grand canonical ensemble,

1. check that the mean number of particles \( N \) and energy \( E \), are given (for both bosons \([N_i = 0, 1, 2, 3, \ldots]\) and fermions \([N_i = 0, 1]\)) by

\[ N = \sum_{i=0}^{\infty} \epsilon_i f_i^{B,F}, \quad E = \sum_{i=0}^{\infty} \epsilon_i f_i^{B,F}, \]  

(2)

where the mean occupation number of the \( i \)-th energy level is given by the Bose-Einstein and the Fermi-Dirac distribution respectively, \( f_i^{B,F} = 1/(e^{\beta(\epsilon_i-\mu)} \pm 1) \).

2 Ideal Bose gas in a 3D box

Consider an ideal Bose gas of \( N \) (non-interacting) bosons in a three-dimensional (3D) cubic box with periodic boundary conditions. The gas is at thermal equilibrium at a temperature \( T \).

2. Show that the expression for the BEC critical temperature

\[ k_B T_c = \frac{2\pi \hbar^2 n^{2/3}}{[g_{3/2}(1)]^{2/3} m} \sim 3.31 \frac{\hbar^2 n^{2/3}}{m} \]  

(3)

is dimensionally correct. Then estimate the critical temperature for condensation \( T_c \) on the basis of this expression for a density \( n \approx 10^{13} - 10^{15} \text{ cm}^{-3} \) and the mass of \(^{87}\text{Rb}\) atoms \((m = 87 \text{ u}, \text{where u is the atomic mass unit})\).
3. Invert numerically the equation which fixes the total number of particles \( N \),
\[
N = \frac{1}{e^{-\beta \mu} - 1} + N \left( \frac{T}{T_c} \right)^{3/2} \int_0^\infty \frac{dx}{x} \frac{x^{3/2}}{e^{-\beta \mu} x^{1/2} - 1} \int_0^\infty \frac{dx}{x} \frac{x^{3/2}}{e^{-\beta \mu} x^{1/2} - 1},
\]
in order to obtain the rescaled chemical potential \( \mu / (k_B T_c) \) as a function of \( N \) and the rescaled temperature \( T / T_c \), where \( T_c \) is the critical temperature for BEC.

4. Plot \( \mu / (k_B T_c) \) as a function of \( T / T_c \) for different values of the total number of particles \( N \) and comment on the results you get.

5. Plot the condensate fraction \( \bar{n}_0 = N_0(N, T / T_c) / N \) and the thermal fraction \( \bar{n}_T = N_T(N, T / T_c) / N \) as a function of \( T / T_c \) for different values of the total number of particles \( N \) and compare the numerical results with the analytical formula valid in the thermodynamic limit.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{plot.png}
\caption{Plot of \( \mu / (k_B T_c) \) and \( \bar{n}_0, \bar{n}_T \) as functions of \( T / T_c \) for different values of \( N \).}
\end{figure}

3 Ideal Bose gas in 2D

6. Show that for a free gas in a box of dimension \( d = 1, 2, 3 \) and volume \( V = L^d \), one has in general that the DoS is given by
\[
\mathcal{N}(\epsilon) = \frac{d G(\epsilon)}{d \epsilon} = \frac{\Omega_d}{2} \left( \frac{2m}{\hbar^2} \right)^{d/2} \epsilon^{d/2 - 1},
\]
where \( \Omega_d = 4\pi (d = 3), 2\pi (d = 2), 1 (d = 1) \).

7. Evaluate the condition to find the critical temperature of an ideal gas in a 2D box (with boundary conditions) and comment why the condition
\[ N_T(\mu \to 0^-, T) \leq N \] can never be satisfied at finite temperature. Can you find a critical temperature \( T_c \)?

8. Do now the same calculation for an ideal gas embedded in a 2D harmonic trap. Evaluate the critical temperature.