A NOTE ON THE ACTION OF THE ABSOLUTE GALOIS GROUP ON DESSINS

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Abstract

We show that the action of the absolute Galois group on dessins d’enfants of given genus \( g \) is faithful, a result that had been previously established for \( g = 0 \) and \( g = 1 \).

1. Belyi surfaces, dessins d’enfants, and the absolute Galois group

One of the fundamental results in Riemann surface theory is the fact that every compact Riemann surface can be thought of as an algebraic curve and every meromorphic function as a rational function. A compact Riemann surface \( C \) is said to be defined (or, better, definable) over some field \( k \) if it is isomorphic (as complex manifold) to the Riemann surface \( C_F \) associate to an algebraic curve \( F(x, y) = 0 \) with coefficients in \( k \). A fundamental problem is then to characterize the definability over some given field. For example, a Riemann surface can be defined over the reals exactly when it has an anticonformal involution.

Belyi proved (see [1]) that when a surface \( C \) can be defined over a number field (thus over \( \mathbb{Q} \)), then it has a meromorphic function \( f \) with only three branching values, what is nowadays called a Belyi function (accordingly \( (C, f) \) is then termed a Belyi pair). The reciprocal statement is also true and follows from previous work of Weil: if \( f : C \rightarrow \hat{\mathbb{C}} \) ramifies only over \( \{0, 1, \infty\} \) then \( C \) and \( f \) are definable over \( \mathbb{Q} \). Definability over number fields is therefore characterized by the existence of Belyi functions.

Part of the interest in this theory comes from the fact that the absolute Galois group \( \text{Gal}(\mathbb{Q}) \) acts on curves defined over \( \mathbb{Q} \). If \( \sigma \in \text{Gal}(\mathbb{Q}) \) and \( C = C_F \), the action is defined by \( \sigma(C) = C_{F^\sigma} \), where \( F^\sigma \) is obtained from \( F \) applying \( \sigma \) to its coefficients. One could thus obtain information about \( \text{Gal}(\mathbb{Q}) \) by understanding this action.

Grothendieck early pointed out the striking fact that Belyi pairs can be described in a very simple combinatorial way, as they correspond exactly to what he called dessins d’enfants (see [2]). A dessin is an embedding of a bipartite graph in a topological surface such that the complement of the graph is a union of simply connected cells. For a Belyi pair \( (C, f) \) the corresponding dessin \( \mathcal{D}_f \) is given by the set \( f^{-1}([0, 1]) \). One can therefore study such an important object as \( \text{Gal}(\mathbb{Q}) \) by making it act on graphs so simple that they look like children’s drawings (dessins...
There are three distinct natural numbers \( M_{\text{enfants}} \), the action being defined by \( \sigma(D_f) = D_{f^g} \). A lot of work has been done in the last years in this direction (see [5], [4], and the references given there).

It is well known that the action of \( \text{Gal}(\overline{\mathbb{Q}}) \) is faithful on dessins of genus 1, see [6], [3]. The action is also faithful on dessins on the sphere, even when restricted to the subclass formed by the trees (see [6]). Nevertheless, as far as we know, there is in the literature no proof of the faithfulness of the action of \( \text{Gal}(\overline{\mathbb{Q}}) \) on genus \( g \) dessins if \( g > 1 \). The aim of this note is to provide such a proof.

2. Faithfulness of the action of \( \text{Gal}(\overline{\mathbb{Q}}) \) on dessins of genus \( g \)

We now show that \( \text{Gal}(\overline{\mathbb{Q}}) \) acts faithfully on genus \( g \) Belyi surfaces \( (g \geq 2) \). In fact it is enough to restrict to hyperelliptic surfaces.

**Theorem 1.** Let \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}) \) be an element of the absolute Galois group, \( \sigma \neq \text{Id} \), and \( a \in \overline{\mathbb{Q}} \) an algebraic number such that \( \sigma(a) \neq a \). Let \( C_n \) \( (n \in \mathbb{N}) \) be the hyperelliptic curve

\[
C_n := \{ y = (x-1)(x-2)\ldots(x-(2g+1))(x-(a+n)) \}.
\]

Then, there is an \( n \) such that \( C_n^\sigma \) is not isomorphic to \( C_n \).

**Proof.** Suppose the theorem false, so that \( C_n^\sigma \simeq C_n \) for all \( n \in \mathbb{N} \). Then, for every \( n \in \mathbb{N} \) there is a M"obius transformation \( M_n \in \text{PSL}_2(\mathbb{C}) \) such that

\[
M_n(\{1,2,\ldots,2g+1,(a+n)\}) = \{1,2,\ldots,2g+1,\sigma(a+n)\}.
\]

We have:

1. \( M_n \in \text{PSL}_2(\mathbb{Q}) \), since it maps three rational points to three rational points.
2. \( M_n(\{1,2,\ldots,2g+1\}) = \{1,2,\ldots,2g+1\} \) by (1), since \( a+n \notin \mathbb{Q} \).
3. \( M_n(a+n) = \sigma(a+n) = \sigma(a)+n \), by (2).
4. There are three distinct natural numbers \( p, q, r \) such that \( M_p = M_q = M_r \). In fact, among all the transformations \( M_n \) there must be infinitely many coincidences, since by (2) \( \{M_n; \ n \in \mathbb{N}\} \) must be a finite set.

We see therefore that

\[
\begin{align*}
M_p(a+p) &= \sigma(a) + p \\
M_p(a+q) &= \sigma(a) + q \\
M_p(a+r) &= \sigma(a) + r
\end{align*}
\]

\[
\Rightarrow
\begin{align*}
M_p(a+p) - \sigma(a) &= p \\
M_p(a+q) - \sigma(a) &= q \\
M_p(a+r) - \sigma(a) &= r
\end{align*}
\]

Let us then consider the M"obius transformation \( M(z) := M_p(a+z) - \sigma(a) \). As \( M(p) = p, M(q) = q \) and \( M(r) = r \), it follows that \( M = \text{Id} \), and then \( M_p(a+z) = z + \sigma(a) \), thus \( M_p(z) = z + \sigma(a) - a \).

Now \( 1 + (\sigma(a) - a) = M_p(1) = l_1 \in \{1,2,\ldots,2g+1\} \), the same applying to \( 2 + (\sigma(a) - a) = M_p(2) = l_2,\ldots,2g+1 \) + (\( \sigma(a) - a ) = M_p(2g+1) = l_{2g+1} \). But then

\[
(\sigma(a) - a) = (l_1 - 1) = (l_2 - 2) = \ldots = (l_{2g+1} - (2g+1))
\]

with \( (l_1 - 1) \geq 0 \) and \( (l_{2g+1} - (2g+1)) \leq 0 \). It follows that \( l_i - i = 0 \) for all \( i \), thus \( l_i = M_p(i) = i \) and \( M_p = \text{Id} \). Then \( \sigma(a) + p = M_p(a+p) = a+p \) and \( \sigma(a) = a \), a contradiction. \( \square \)
References


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