Moduli of Riemann surfaces with symmetry

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To Murray Macbeath on the occasion of his retirement

The moduli space $\mathcal{M}_g$ of Riemann surfaces with genus $g \geq 2$ contains an important subset corresponding to surfaces admitting non-trivial automorphisms. In this paper, we study certain irreducible subvarieties $\mathcal{M}_g(G)$ of this singular set, which are characterised by the specification of a finite group $G$ of mapping-classes whose action on a surface $S$ is fixed geometrically. In the special case when the quotient surface $S/G$ is the sphere, we describe a holomorphic parameter function $\lambda$ which extends the classical $\lambda$-function of elliptic modular theory, and which induces a birational isomorphism between the normalisation of $\mathcal{M}_g(G)$ and a certain naturally defined quotient of a configuration space $\mathcal{C}^n - \Delta$ where $\Delta$ is the discriminant set \{${z_i = z_j, \text{ for some } i \neq j}$\}. Thus $\mathcal{M}_g(G)$ is always a unirational variety. We also show that in general $\mathcal{M}_g(G)$ is distinct from its normalisation, and construct a (coarse) modular family of $G$-symmetric surfaces over the latter space.

1. Teichmüller spaces and modular groups

First we introduce some of the necessary formalism. Let $H_0$ be a subgroup of the group $\text{Aut}(S_0)$ of automorphisms of a closed surface $S_0$ of genus $g \geq 2$; by a famous theorem of Hurwitz (see e.g. [19]), $\text{Aut}(S_0)$ is finite of order at most $84(g - 1)$. We shall later concentrate on the case where the quotient
surface $S_0/H_0$ is $\mathbb{P}^1$, the Riemann sphere, but results in §1 and §2 apply without this restriction.

**Definition.** A Riemann Surface with $H_0$-symmetry is a pair $(S, H)$ comprising a Riemann surface $S$ with $H < \text{Aut}(S)$ such that $(S_0, H_0)$ and $(S, H)$ are topologically conjugate by some homeomorphism $\theta$: $S_0 \to S$.

Two surfaces $(S, H), (S', H')$ with $H_0$-symmetry are $H_0$-isomorphic if there is a biholomorphic mapping $\phi$: $S \to S'$ such that $H' = \phi H \phi^{-1}$.

**Notation.** An $H_0$-isomorphism class is denoted by $\{S, H\}$ and the set of all $H_0$-isomorphism classes of surfaces with $H_0$-symmetry is denoted by $\mathcal{M}_g(H_0)$.

We shall also need to consider the weaker equivalence relation of (non-equivariant) isomorphism for surfaces $(S, H), (S', H')$ with $H_0$-symmetry; here there must be a biholomorphic mapping $\phi$: $S \to S'$ as before, but it is no longer required to satisfy the condition $H' = \phi H \phi^{-1}$. We shall denote by $\widetilde{\mathcal{M}}_g(H_0)$ the set of all isomorphism classes of surfaces with $H_0$-symmetry.

There is a natural surjection $\mathcal{M}_g(H_0) \to \mathcal{M}_g(H_0)$ between these two sets. Our primary purpose is to provide complex analytic structures for them which make this mapping a morphism of analytic spaces. Our approach rests on well-known results of Teichmüller theory which we now discuss briefly. Good references for the facts we need are [9], [22]. More details of our methods are given in earlier papers [14, 20].

Let $T_g$ be the Teichmüller space of $S_0$. A point $t \in T_g$ is an equivalence class $[S, \theta]$, where $\theta: S_0 \to S$ is a marking homeomorphism, and two marked pairs $(S, \theta), (S', \theta')$ are equivalent iff there is a biholomorphic $f$: $S \to S'$ such that $\theta'$ is isotopic to $f \circ \theta$.

If $b = \{b_1, \ldots, b_n\}$ is a finite subset of $S_0$, and $S_0^* = S_0 - b$ denotes the surface punctured at $b$, then the (stronger) equivalence relation obtained by requiring the isotopy between $\theta'$ and $f \circ \theta$ to fix the points of $b$ determines the Teichmüller space $T_{g,n}$ of $S_0^*$, $n \geq 1$.

The group of mapping classes $\text{Mod}(S_0)$, viewed as the path components of the group of homeomorphisms of $S_0$, is denoted $\text{Mod}_g$ if $S_0$ has genus $g$ (or $\text{Mod}_{g,n}$ for $S_0^*$). This group operates on $T_g$ (or on $T_{g,n}$) by the rule

$$[S, \theta] \mapsto [S, \theta \circ f]$$
By fundamental results of Bers [3], there is a canonical representation of each \( T_{g,n} \) as a bounded domain in some \( \mathbb{C}^N \), with \( N = 3g - 3 + n \). Furthermore, the action of \( \text{Mod}_{g,n} \) is by holomorphic isomorphisms and properly discontinuous [18], [20].

We shall regard a subgroup \( H_0 \subseteq \text{Aut}(S_0) \) as tantamount to a subgroup of \( \text{Mod}(S_0) \), since by a theorem of Hurwitz an automorphism of \( S_0 \) that is homotopic to the identity must be trivial. By a result which goes back to W. Fenchel and J. Nielsen, the fixed point set in \( T_g \) of any such finite group \( G \subset \text{Mod}(S_0) \) is a (complex) submanifold denoted by \( T_g(G) \).\(^1\) In the present terminology it was reformulated in [14] as follows.

**Theorem A.** \( T_g(H_0) \) is the set of Teichmüller points \([S, \theta]\) such that \( S \) possesses a group of automorphisms \( H \) conjugate to \( H_0 \) by means of the homeomorphism \( \theta: S_0 \rightarrow S \).

Because the action of the modular group on \( T_g \) is properly discontinuous, the quotient moduli space \( \mathcal{M}_g \) carries an induced structure of complex analytic \( \mathbb{V} \)-manifold, for which the canonical projection map \( p: T_g \rightarrow \mathcal{M}_g \) is holomorphic. In fact \( \mathcal{M}_g \) is a projective variety; it is worth noting that the \( \Delta \)-functions which we describe later fit in naturally with the projective embedding originally constructed by Baily [1] using Jacobi varieties and the Lefschetz embedding theorem.

**Corollary.** \( \mathcal{M}_g(H_0) \) is the image of \( T_g(H_0) \) under the projection \( p \).

The submanifold \( T_g(H_0) \) is itself a Teichmüller space. To see this, let the quotient surface \( R_0 = S_0/H_0 \) have genus \( \gamma \), let \( b = \{b_1, \ldots, b_r\} \) be the point set over which the projection \( S_0 \rightarrow R_0 \) is ramified and denote by \( T_{\gamma,r} \) the Teichmüller space of the punctured surface \( R_0^* = R_0 - b \).

For each \([S, \theta] \in T_g(H_0)\), write \( R \) for the quotient surface \( S/H \) and \( R^* \) for the corresponding unramified subsurface. Then \( \theta: S_0 \rightarrow S \) induces a homeomorphism \( \theta^*: R_0^* \rightarrow R^* \), which defines a rule

\[
[S, \theta] \mapsto [R^*, \theta^*].
\]

At the level of Teichmüller spaces, this is a bijection.

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\(^1\) The fact that \( T_g(G) \) is non-empty for all finite \( G \) was proved by S. Kerckhoff [17].
Theorem B. The spaces \( T_g(H_0) \) and \( T_{\gamma,r} \) are biholomorphically equivalent via the mapping \( \psi \).

For a proof, see [18], [14], [23].

Not every element in \( Mod_g \) stabilizes \( T_g(H_0) \). The modular group permutes the various finite subgroups \( H_0 \) by conjugation and the relevant group for our purposes is the relative modular group with respect to \( H_0 \), which is defined as the subgroup of those mapping classes that do stabilise \( T_g(H_0) \); this is the normaliser of \( H_0 \) in \( Mod_g \) (see[20]). We denote it by \( Mod_g(H_0) \).

For each \( [S, \theta] \in T_g(H_0) \) with a marked symmetry group \( H = \theta H_0 \theta^{-1} \), the rule \( [S, \theta] \to \{ S, H \} \) defines a mapping from \( T_g(H_0) \) into \( \bar{M}_g(H_0) \) which we shall denote by \( \pi_1 \). This map is clearly surjective.

Let \( f \) be an \( H_0 \)-equivariant homeomorphism of \( S_0 \) representing an element \( f \) of \( \bar{M}_g(H_0) \). Then \( f([S, \theta]) = [S, \theta \circ f] \) has marked symmetry group \( H_f \), obtained \emph{via} \( H_f = (\theta \circ f) H_0(\theta \circ f)^{-1} = \theta H_0 \theta^{-1} \). Notice that the underlying Riemann surface \( S \) and its automorphism group \( H \) are unchanged: the change of marking by \( f \) induces a complementary change of marking for \( H \). This implies that the images under \( \pi_1 \) of \( [S, \theta] \) and \( f([S, \theta]) \) coincide in \( \bar{M}_g(H_0) \).

Suppose now that we have two pairs \( (S_1, H_1), (S_2, H_2) \) of surfaces with \( H_0 \)-symmetry, related by a biholomorphic isomorphism \( \phi : S_1 \to S_2 \) such that \( H_2 = \phi H_1 \phi^{-1} \). Choose two markings \( \theta_j : S_0 \to S_j \), \( j = 1, 2 \), so that each \( [S_j, H_j] \) is a Teichmüller point in \( T_g(H_0) \) lying over \( \{ S_j, H_j \} \). Then there is a homeomorphism \( f : S_0 \to S_0 \) making the diagram

\[
\begin{array}{ccc}
S_0 & \xrightarrow{\theta_1} & S_1 \\
\downarrow f & & \downarrow \phi \\
S_0 & \xrightarrow{\theta_2} & S_2
\end{array}
\]

commute and compatible with \( H_0(= \theta_2^{-1} H_j \theta_2) \). Therefore \( f \) determines an element of \( Mod_g(H_0) \) and we have proved the following statement.

Proposition 1. The mapping \( \pi_1 : T_g(H_0) \to \bar{M}_g(H_0) \) induces a natural bijection between the quotient space \( T_g(H_0)/Mod_g(H_0) \) and \( \bar{M}_g(H_0) \).

Since \( T_g(H_0) \cong T_{\gamma,r} \) is a bounded domain in \( \mathbb{C}^m \) where \( m = 3\gamma - 3 + r \), it follows that \( \bar{M}_g(H_0) \) is a complex \( V \)-manifold of dimension \( m \).
2. The relationship between $\widetilde{M}_g(H_0)$ and $\mathcal{M}_g(H_0)$.

The first aim of this section is to prove that $\widetilde{M}_g(H_0)$ is the normalisation of $\mathcal{M}_g(H_0)$. We shall use [11] as a basic reference for analytic spaces.

**Theorem 1.** $\mathcal{M}_g(H_0)$ is an irreducible subvariety of $\mathcal{M}_g$ and $\widetilde{M}_g(H_0)$ is its normalisation.

**Proof.** Because $\text{Mod}_g(H_0)$ acts discontinuously with finite isotropy groups on $T_g(H_0) \cong T_{g, \tau}$, a domain in $\mathbb{C}^m$, it follows from a theorem of Cartan [4] that $\widetilde{M}_g(H_0)$ is a normal complex space. Also, by the discontinuity of $\text{Mod}_g$ on $T_g$, the family of submanifolds $\{h(T_g(H_0)), h \in \text{Mod}_g\}$ is locally finite, that is each point of $T_g(H_0)$ has a neighbourhood in $T_g$ intersecting only finitely many distinct subvarieties $h(T_g(H_0))$.

We have already defined the natural mapping $\pi: \widetilde{M}_g(H_0) \to \mathcal{M}_g$ whose image is precisely $\mathcal{M}_g(H_0)$. Thus if we check that:

1. $\pi$ is closed,
2. $\pi$ has finite fibres,
3. $\pi$ is injective outside a proper subvariety,

then by the Proper Mapping Theorem and the definition of normalisation the theorem will be proved.

Let us prove (2) and (3). The diagram below summarises the situation; the map $\pi_2 = \pi \circ \pi_1: T_g(H_0) \to \mathcal{M}_g(H_0)$ is the restriction of $p$ to $T_g(H_0)$.

\[
\begin{array}{ccc}
T_g(H_0) & \xrightarrow{\pi_1} & T_g \\
\downarrow & & \downarrow \pi \\
\widetilde{M}_g(H_0) & \xrightarrow{\pi} & \mathcal{M}_g(H_0) & \xrightarrow{\pi_2} & \mathcal{M}_g
\end{array}
\]

Two points in $T_g(H_0)$ with the same image in $\mathcal{M}_g(H_0)$ are of the form $[S, \theta]$ and $[S, \theta \circ h]$ with $h \in \text{Mod}_g$. By Theorem A, $H = \theta H_0 \theta^{-1}$ and $H' = (\theta \circ h)H_0(\theta \circ h)^{-1}$ are both subgroups of $\text{Aut}(S)$. Now, if $[S, \theta]$ and $[S, \theta \circ h]$ have different images in $\widetilde{M}_g(H_0)$, then $h \notin \text{Mod}_g(H_0)$ and so $hH_0h^{-1} \neq H_0$. Hence necessarily $H \neq H'$. Since $\text{Aut}(S)$ is finite, there are only finitely many possibilities for $[S, \theta \circ h]$, which proves (2).

This argument also shows that $\pi$ fails to be injective only on the $\pi_1$-image of intersections $T_g(H_0) \cap h(T_g(H_0))$ with $h \in \text{Mod}_g - \text{Mod}_g(H_0)$. By the local finiteness this is a subvariety of $\widetilde{M}_g(H_0)$, which proves (3).
For completeness we sketch the elementary property (1). Referring to the diagram above, it is sufficient to prove that if \( C \) is a closed subset of \( T_g(H_0) \) then \( \pi_2(C) \) is closed in \( \mathcal{M}_g(H_0) \) or, equivalently, that the union of all \( h(C) \), \( h \in \text{Mod}_g \), is closed in \( T_g \). Suppose that \( y = \lim h_n(x_n) \) with \( x_n \in C \) and \( h_n \in \text{Mod}_g \). Taking \( N_y \) a small enough open set in \( T_g \) containing \( y \) such that \( N_y \) intersects only finitely many sets \( h(C) \), there is then a single set \( h_0(C) \) which contains an infinite subsequence of the \( \{h_n(x_n)\} \). Thus we have a sequence of points \( x'_n \in C \) with \( h_0(x'_n) \to y \). But \( T_g(H_0) \) is closed in \( T_g \) and \( h_0 \) is an isometry in the Teichmüller metric, so it follows that \( x'_n \to x \in C \). This completes the verification of the property (1).

We next address the question whether \( \overline{\mathcal{M}}_g(H_0) \) is biholomorphic to \( \mathcal{M}_g(H_0) \). From the proof of the theorem we can see that these spaces are different if and only if there is a surface \( S \) whose automorphism group contains two subgroups \( H, H' \) that are conjugate topologically but not holomorphically. This situation occurs, for instance, when there is a surface \( S \), which admits a larger group \( G \) of automorphisms containing a pair of (conjugate) subgroups \( H, H' \) such that \( (H, H') = K \) is a proper subgroup of \( G \) with \( H, H' \) not conjugate in \( K \). Usually a deformation of \( S \) may then be constructed which preserves the \( K \)-symmetry but destroys the \( G \)-symmetry. Examples are readily produced using the fact that for any finite group \( G \) there exist Riemann surfaces \( S \) with \( G \) as a group of automorphisms and such that the quotient surface \( S/G \) has arbitrarily given genus \( \gamma \); see for instance [12]. An elementary example of this type is given later in this section (example 1).

Provided that the Teichmüller space \( T_g(K) \) is not a point (the case \( T_{0,3} \) ) and is not in the small list of types for which there is an isomorphism between Teichmüller spaces of surfaces with different signatures, we may conclude that the space \( T_g(G) \) is properly contained in \( T_g(K) \) for any proper overgroup \( G > K \). This list is as follows (see [22] p.129 for details):

\[
T_{0,6} \cong T_{2,0}, \quad T_{0,5} \cong T_{1,2}, \quad T_{0,4} \cong T_{1,1}.
\]

The implication is that in general the locus of points \( [S, \theta] \in T_g(K) \), with \( S \) admitting two automorphism groups \( \theta H \theta^{-1} \) and \( \theta H' \theta^{-1} \) which are conjugate only in some larger group than \( \theta K \theta^{-1} \), forms an analytic subset \( Z \) of strictly lower dimension. Therefore the restriction of the mapping \( \pi : \overline{\mathcal{M}}_g(H) \to \mathcal{M}_g(H) \) to the \( \pi_1 \)-image of \( T_g(K) \) is not injective since outside \( \pi_1(Z) \) one has \( \pi([S, H]) = \pi([S, H']) \). Thus \( \pi \) is not biholomorphic, and the variety \( \mathcal{M}_g(H) \) is non-normal at all points in the image of \( \pi_1(T_g(K) - Z) \).
Furthermore, using elementary facts on analytic spaces, we can conclude that since this subset of the non-normal points of $\mathcal{M}_g(H)$ is Zariski-open in $\mathcal{M}_g(K)$, and therefore dense, the whole subvariety $\mathcal{M}_g(K) = \pi_1(T_g(K)$ is non-normal because the non-normal set is necessarily closed ([11], p.128).

These arguments prove the following result.

**Theorem 2.** The modular subvariety $\mathcal{M}_g(H_0)$ is in general distinct from its normalisation $\overline{\mathcal{M}}_g(H_0)$.

As illustration, we give two examples.

**Example 1.** Let $F_{2p}$ be the compact (Fermat) Riemann surface with affine algebraic equation

$$x^{2p} + y^{2p} = 1,$$

let $H$ (respectively $H'$) be the cyclic group generated by the involution $(x, y) \to (-x, y)$ (respectively by $(x, y) \to (x, -y)$) and let $K = \langle H, H' \rangle$. Then $H$ and $H'$ are not conjugate in $K$ but in $G = \text{Aut}(F_{2p})$ they are conjugate by the automorphism $\alpha(x, y) = (y, x)$. Now $F_{2p}/K$ has genus $> 2$; in fact $F_{2p}/K$ is isomorphic to the surface $F_p$ with equation $x^p + y^p = 1$, the isomorphism being given in affine coordinates by $\phi(x, y) = (x^2, y^2)$; $F_p$ has genus $(p - 1)(p - 2)/2$ which is $> 2$ for $p \geq 4$.

Thus, by the discussion preceding theorem 2, the modular subvariety $\mathcal{M}_g(H), \ g = (p - 1)(2p - 1)$, is not normal; in fact the point representing the Fermat surface $F_{2p}$ is a non-normal point of this modular subvariety.

On the other hand, for certain types of surface with automorphism, the modular subvariety $\mathcal{M}_g(H)$ is itself normal.

**Example 2.** Let $S$ be a hyperelliptic surface of genus $g$, with $J : S \to S$ the hyperelliptic involution. Since $J$ is the unique automorphism of $S$ with order 2 having quotient $S/\langle J \rangle \equiv \mathbb{P}^1$, we obtain $\overline{\mathcal{M}}_g(\langle J \rangle) = \mathcal{M}_g(\langle J \rangle)$.

**Remark.** Since $\mathcal{M}_g$ is a projective variety, the G.A.G.A. Principle implies that our complex-analytic results remain valid within the framework of complex algebraic geometry. Thus $\mathcal{M}_g(G)$ is also an irreducible algebraic subvariety of $\mathcal{M}_g$ by Chow's Theorem ([11] p.184). Furthermore, since the algebraic normalisation of a projective variety is again projective ([13] p.232) and therefore analytic, it follows from the uniqueness of the normalisation ([11] p.164) that $\overline{\mathcal{M}}_g(G)$ is also the algebraic normalisation of $\mathcal{M}_g(G)$. 
3. The case of tori: Legendre’s modular function.

We review the classical theory of moduli for elliptic curves from the point of view developed in the previous section. In genus 1, the Teichmüller space $T_1$ is the upper half plane $U$: any Riemann surface of genus 1 may be expressed as a complex torus $E = E_r = \mathbb{C}/\Lambda(\tau)$ with $\Lambda(\tau) = \mathbb{Z} + \mathbb{Z}\tau$ a lattice subgroup of the additive group $\mathbb{C}$ and $\tau \in U$. There is a standard involutory automorphism $J : E \to E$, given by the symmetry $z \to -z$ of $\mathbb{C}$ and so, writing $H = \langle J \rangle$, we have that $T_1(H) = T_1$. The quotient $E/H$ is the projective line $\mathbb{P}^1$, with four ramification points $a_1, \ldots, a_4$ corresponding to the four fixed points of $J$ (which are the points of order 2, the orbits of $0, \frac{1}{2}, \frac{1+\sqrt{3}}{2}$, and $\frac{1}{2}$ under $\Lambda(\tau)$).

Let the orbit of the origin 0 be chosen as a base point of $E$ and write $T_{1,1}$ for $T(E - 0)$: this procedure renders the (flat) homogeneous space $E$ into a hyperbolic surface, thereby placing the theory of moduli for $E$ within the framework of Teichmüller spaces. Theorem B now captures the identification of T-spaces, $T_{1,1} \cong T_{0,4}$, in our earlier list.

This space may be identified with the upper half-plane $U$ by associating to $\tau \in U$ the Teichmüller pair $[E_\tau, f_\tau]$, $f_\tau : E_i \to E_\tau$, where $E_i = \mathbb{C}/\Lambda(i)$ has been chosen as reference surface and $f_\tau$ is the projection of the real linear homeomorphism $L_\tau : \mathbb{C} \to \mathbb{C}$ which sends 1, i to 1, $\tau$ respectively (see e.g. [22] 2.1.8).

Similarly the fact that $T_{1,1}(H)$ is the whole of $T_{1,1}$ implies, by the definition of relative modular group given in §1, that $\text{Mod}_1(H)$ is the modular group of genus 1, $\text{SL}(2, \mathbb{Z})$, so we have

$$\mathcal{M}_1(J) \equiv \mathcal{M}_1 \equiv U/\text{SL}(2, \mathbb{Z}).$$

Here $\text{SL}(2, \mathbb{Z})$ acts by $A \cdot \tau = \frac{a\tau + b}{c\tau + d}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We identify $\tau$ with $[E_\tau, f_\tau]$ and $A$ with the homeomorphism $f_A : E_i \to E_i$ characterised by the real-linear map $L_A(1) = ci + d$, $L_A(i) = ai + b$. Then the Teichmüller modular group acts by the rule

$$f_A \cdot (E_\tau, f_\tau) = (E_\tau, f_\tau \circ f_A) = (E_A \cdot \tau, h_A \circ f_\tau \circ f_A)$$

where $h_A : E_\tau \to E_A \cdot \tau$ is the isomorphism induced by $h_A(z) = (c\tau + d)^{-1}z$. To see that this is a genuine group action, one checks directly that $h_A \circ f_\tau \circ f_A$ is just $f_{A \cdot \tau}$, so we have $f_A \cdot (E_\tau, f_\tau) = (E_A \cdot \tau, f_{A \cdot \tau})$ as it should be.
Next we focus attention on the level-2 congruence subgroup \( \Gamma(2) \), comprising matrices \( A \equiv \text{Id} \pmod{2} \) in \( SL_2(\mathbb{Z}) \). The involution \( J \) corresponds to the central element \( \left( \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right) \) of \( \Gamma(2) \), which fixes every point of \( U \). The quotient group \( P\Gamma(2) \) acts faithfully on \( U \) and (see for instance [5], [16]) this action is free and discontinuous. In classical vein, we define a complex function \( \lambda \) on \( U \) by the rule

\[
\lambda(\tau) = \{ \varphi(a_1), \varphi(a_2), \varphi(a_3), \varphi(a_4) \},
\]

where \( \varphi(z) = \varphi(\tau)(z) \) is the Weierstrass \( \varphi \)-function of the lattice \( \Lambda(\tau) \) and \( \{-,\tau;-,\} \) denotes cross-ratio; \( \lambda \) is the Legendre modular function, which is automorphic with respect to \( P\Gamma(2) \) and induces an isomorphism

\[
\lambda: U/P\Gamma(2) \to \mathbb{C} - \{0,1\}.
\]

The famous modular invariant \( j(\tau) \) may then be written as an invariant (degree 6) rational function of \( \lambda \).

We shall need the following description of this classical theory in terms of the universal family \( E \) of tori over \( U \); a brief account appears in [26]. This is a fibre space \( E = (U \times \mathbb{C})/\mathbb{Z}^2 \) over \( U \) where \( \mathbb{Z}^2 \) acts on \( U \times C \) by \( (n,m) \cdot (\tau; z) = (\tau; z + n + m \tau) \), so that the fibre over \( \tau \) is precisely \( E_\tau \).

Corresponding to the four fixed points of \( J \), we have the following four holomorphic sections of the family \( E \to U \),

\[
s_1(\tau) = 0, \quad s_2(\tau) = \frac{\tau}{2}, \quad s_3(\tau) = \frac{1 + \tau}{2}, \quad s_4(\tau) = \frac{1}{2}.
\]

By normalising the \( \varphi \)-function we obtain a meromorphic function \( x(\tau, z) \) on \( E \) which when restricted to each fibre gives rise to a function \( x_\tau: E_\tau \to \mathbb{P}^1 \), having these four points as branch points and with corresponding branch values

\[
x(\tau, s_1(\tau)) = \infty, \quad x(\tau, s_2(\tau)) = 1, \quad x(\tau, s_3(\tau)) = 0, \quad x(\tau, s_4(\tau)) = \lambda(\tau).
\]

Finally, the congruence group \( \Gamma(2) \) can be characterised as the group of matrices \( A \) such that the corresponding mapping classes \( f_A \) introduced above preserve each of these four points; and \( SL(2,\mathbb{Z})/\Gamma(2) \) is isomorphic to the subgroup stabilising \( s_1 \) of the symmetric group \( \Sigma_4 \) which permutes the \( \{s_j\} \). This description will become relevant later on.
References


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