Russian Doll Renormalization Group and Superconductivity

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The Kosterlitz-Thouless diagram exhibits a rich behavior in the RG flows

1. Introduction: RG fixed points and cyclic limits

The Kosterlitz-Thouless diagram exhibits a rich behavior in the RG flows

2D XY spin systems

- IR fixed points
- UV fixed points
- Cyclic regime

2. Extended BCS Hamiltonian

The Hamiltonian for the study of a superconducting system with $N$-levels is given by:

$$H = \sum_{i=1}^{N} \frac{\varepsilon_i}{2} c_{i,i \sigma}^\dagger c_{i,i \sigma} - \sum_{i,j=1}^{N} V_{ij} c_{i,i \uparrow}^\dagger c_{j,j \downarrow} - c_{j,j \uparrow}^\dagger c_{i,i \downarrow}^\dagger V_{ij} = V_{ji}$$

$$b_i^\dagger = c_{i,i \uparrow}^\dagger c_{i,i \downarrow}^\dagger$$

PAIR OPERATORS

Single electrons decouple

$$H c_{i,i \uparrow}^\dagger |\psi\rangle = \frac{\varepsilon_i}{2} c_{i,i \uparrow}^\dagger |\psi\rangle + c_{i,i \uparrow}^\dagger H |\psi\rangle$$

We consider just a hamiltonian for pairs

$$V_{ij} = \begin{cases} G + i\Theta & \text{if } \varepsilon_i > \varepsilon_j \\ G & \text{if } \varepsilon_i = \varepsilon_j \\ G - i\Theta & \text{if } \varepsilon_i < \varepsilon_j \end{cases}$$

The extended BCS Hamiltonian:

$$H_{BCS} = \sum_{i=1}^{N} (\varepsilon_i - G) b_i^\dagger b_i - (G + i\Theta) \sum_{i>j=1}^{N} b_i^\dagger b_j - (G - i\Theta) \sum_{i<j=1}^{N} b_i^\dagger b_j$$
3. Mean-field solution

The BCS variational ansatz:

\[
|\psi_{BCS}\rangle = \prod_{i=1}^{N} (u_i + v_i b_i^\dagger)|0\rangle
\]

yields

\[
u_i^2 = \frac{1}{2} \left( 1 + \frac{\xi_i}{\sqrt{\xi_i^2 + \Delta_i^2}} \right)
\]

and

\[
v_i^2 = \frac{1}{2} e^{i\theta_i} \left( 1 - \frac{\xi_i}{\sqrt{\xi_i^2 + \Delta_i^2}} \right)
\]

\[
\Delta_i = \sum_j \frac{V_{ij} \Delta_j}{\sqrt{\xi_j^2 + \Delta_j^2}}
\]

where

\[
\Delta_i \equiv \Delta_i e^{i\theta_i}
\]

Taking the continuum limit:

\[
\{ \Delta_i \to \Delta(\epsilon) \}
\]

\[
\{ \phi_i \to \phi(\epsilon) \}
\]

\[
\tilde{\Delta}(\epsilon) = g \int_{-\omega}^{\omega} \frac{d\epsilon'}{2} \frac{\Delta(\epsilon)}{\sqrt{\epsilon'^2 + \Delta^2}} + i\theta \int_{-\omega}^{\omega} \frac{d\epsilon'}{2} \frac{\Delta(\epsilon)}{\sqrt{\epsilon'^2 + \Delta^2}}
\]

\[
\Delta(\epsilon) = \Delta = \text{const.}
\]

\[
\frac{d\phi}{d\epsilon} = \frac{\theta}{\sqrt{\epsilon^2 + \Delta^2}}
\]

\[
G = g\delta
\]

\[
\Theta = \theta\delta
\]
General solution to the gap equation:

$$\Delta_n = \frac{\omega}{\sinh t_n} \quad t_n = t_0 + \frac{n\pi}{\theta} \quad n = 0, 1, 2, ..., \infty$$

$$\tan(\theta t_n) = \frac{\theta}{g}$$

Infinite number of solutions

Spectrum

- $E = 0$
- $E_2(\Delta_2)$
- $E_1(\Delta_1)$
- $E_0(\Delta_0)$

In the low energy regime: \(\Delta_n \ll \omega \quad \Rightarrow \quad n \gg \frac{\theta}{\pi}\)

the condensation energies:

$$E_C^{(n)} = -\frac{\Delta_n^2}{8\delta}$$

$$\Delta_n - 2N\delta \exp(-t_0 - n\pi / \theta)$$

Scaling behavior

$$E_C^{(n)} = -\frac{1}{2} N^2\delta \exp(-2t_0 - 2n\pi / \theta)$$
In a discrete system the RG eliminates energy levels

\[ H_N \rightarrow H_{N-1} \rightarrow H_{N-2} \]

An effective hamiltonian can be constructed by considering the virtual processes involving the N-th level

Effective coupling:

\[ V_{ij}^{(N-1)} = V_{ij}^{(N)} + \frac{1}{2} V_{iN}^{(N)} V_{Nj}^{(N)} \left( \frac{1}{\xi_N - \xi_i} + \frac{1}{\xi_N - \xi_j} \right) \]

\[ V_{ij} = V_{ji}^* \quad \xi_i = \varepsilon_i - \mu - V_{ii} \]
In our problem:
\[
\begin{align*}
V_{ij} &= G \pm i\Theta \\
\Theta &= \theta \delta \\
\epsilon_N &= \epsilon_1 \approx \omega = N\delta \gg \epsilon_i
\end{align*}
\]

\[
g_{N-1} = g_N + \frac{1}{N} \left( g_N^2 + \theta_N^2 \right) \\
\theta_{N-1} = \theta_N
\]

RG invariant

In the large-N limit we define a RG scale:
\[
s \equiv \log \frac{N_0}{N}
\]

Beta function:
\[
\frac{dg}{ds} = g^2 + \theta^2 \quad \theta = \text{const}
\]

RG solution:
\[
g(s) = \theta \tan \left[ \theta s + \tan^{-1} \frac{g_0}{\theta} \right]
\]
\[
g_0 = g(N_0)
\]

Cyclic behavior of the RG:
\[
g(s + \lambda) = g(s) \iff g(e^{-\lambda}N) = g(N) \quad \lambda \equiv \frac{\pi}{\theta}
\]
The cyclicity is shown in the spectrum:

- After a cycle $\lambda$ we recover the same coupling
- Two systems with sizes $N$ and $N’ = Ne^{-\lambda}$ and the same couplings have the same spectrum

$$\{E(g,\theta, e^{-\lambda}N)\} = \{E(g,\theta, N)\} \quad g, \theta \text{ fixed}$$

Agrees with mean-field solution

One Cooper pair for fixed couplings and different sizes
5. Synchronicity of the mean-field and the RG

Running coupling constant: \[ g(s) = \theta \tan \left[ \theta \left( s + \tan^{-1} \frac{g_0}{\theta} \right) \right] \]

What does it happen at points where \( g(s) \to \infty \)?

Using the MF solution: \[ \tan(\theta t_n) = \frac{\theta}{g} \]
\[ \Delta_n = \frac{\omega}{\sinh t_n} \]

\[ g(s) = \theta \tan \left[ \theta (s - t_n) + \frac{\pi}{2} \right] \]

\( g(s) \to \infty \) for scales equal to mean-field solutions; a condensate disappears from the spectrum

Russian-doll Renormalization Group

\[ \Delta_0(g = +\infty) = +\infty \]
\[ \Delta_{n+1}(g = +\infty) = \Delta_n(g = -\infty) \]
\[ \Leftrightarrow \left\{ \begin{array}{c} E_c^{(0)}(g = +\infty) = -\infty \\ E_c^{(n+1)}(g = +\infty) = E_c^{(n)}(g = -\infty) \end{array} \right\} \]

\[ g_0 = 1 \]
\[ \theta = 12 \]
6. Conclusions and prospects

1. We have presented a complex extension \((g \pm i\theta)\) of the standard BCS Hamiltonian.

2. The Mean-Field solution yields an infinite number of condensates, related by a scaling transformation.

3. The coupling constant shows a cyclic behavior under RG transformations, \(\lambda \equiv \pi /\theta\).

4. This Russian-doll behavior of the RG is intimately related to the existence of the condensates.

PROSPECTS:

- The parameter \(\theta\) could provide for a mechanism to increase the value of the gap.

- The existence of multiple gaps would be revealed by a set of critical temperatures following: \(\Delta_n(0)/T_{c,n} \approx 3.52\)

- Models as XXZ spin chain or sine-Gordon are susceptible of presenting limit cycles in their RG flow, which could be understood from a string formulation.