Smoothing and Decay Estimates
for Nonlinear Diffusion Equations

Equations of Porous Medium Type

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Preface

This text is concerned with quantitative aspects of the theory of nonlinear diffusion equations. These equations can be seen as nonlinear variations of the classical heat equation, the well-known paradigm to explain diffusion, and appear as mathematical models in different branches of Physics, Chemistry, Biology and Engineering. They are also relevant in Differential Geometry and Relativistic Physics. Much of the modern theory of such equations is based on estimates and functional analysis. Indeed, Nonlinear Functional Analysis is a quite active branch of Mathematics, and a large part of its activity is aimed at providing tools for solving the equations originated in scientific disciplines like the above-mentioned.

We concentrate on a class of equations with nonlinearities of power type that lead to degenerate or singular parabolicity and we gather collectively under the name “equations of porous medium type”. Particular cases are the Porous Medium Equation, the Fast Diffusion Equation and the evolution $p$-Laplacian Equation. These equations have a wide number of applications, ranging from plasma physics to filtration in porous media, thin films, Riemannian geometry and many others. And they have at the same time served as a testing ground for the development of new methods of analytical investigation, since they offer a variety of surprising phenomena that strongly deviate from the heat equation standard. Among those phenomena we count free boundaries, limited regularity, mass loss, and extinction or quenching, to quote a few.

The aim of the present work is obtaining sharp a priori estimates and decay rates for general classes of solutions of those equations in terms of estimates of particular problems. The estimates will be building blocks in understanding the qualitative theory, the decay rates should pave the way to the fine study of asymptotics. Basic tools are results of symmetrization and mass concentration comparison, combined with scaling properties; all of this reduces the problem to getting a detailed knowledge of special solutions using worst-case strategies. The functional setting consists of Lebesgue and Marcinkiewicz spaces, and our final aim is to get a deeper knowledge of the evolution semigroup generated by the equation. We obtain optimal estimates with best constants. Many technically relevant questions are presented and analyzed
in detail, like the question of strong smoothing effects versus weak smoothing effects. The end result combines a number of properties that extend the linear parabolic theory with an array of peculiar phenomena. As a summary, a systematic picture of the most relevant phenomena is obtained for the equations under study, including time decay, smoothing, extinction in finite time, and delayed regularity.

Being based on estimates, this is essentially a book about mathematical inequalities and their impact on the theory. A classic in that respect is no doubt the treatise “Inequalities” by G. Hardy, J. E. Littlewood and G. Pólya, [HLP64]. Another source of motivation is the famous line of inequalities known collectively as Sobolev inequalities, that permeate the study of nonlinear PDEs since the middle of the 20th century. We recall that in mathematics an inequality is simply a statement about the relative size or order of two objects. Our inequalities determine or control the behaviour of nonlinear diffusion semigroups in terms of data and parameters. That sums up our game in simple terms.

The present text contains results taken from papers of the author and collaborators on the theory of nonlinear diffusion, and also the main progress due to other authors. Together with the monograph [V06] in which we develop the mathematical theory of the PME, the surveys [Va03], [Va04] on asymptotic behaviour, and the text co-authored with V. Galaktionov on a Dynamical Systems approach to nonlinear PDE evolution problems, [GV03], it represents an effort of the author to present to a wide audience a substantial part of the work involving the PME/FDE that has been developed in the last decades, and as a support for the work that continues to be done nowadays in new directions. The book contains a fair amount of new results and open problems; actually, we feel that further ideas and understanding are still needed in this area, and even more in its many interactions with other subjects in the wide world of nonlinear PDEs.

Acknowledgments

This text is the result of many years of thinking on the topics of nonlinear semigroups, bounds and asymptotics. It is a pleasure to mention some of the people who made possible this particular journey through the kingdom of Nonlinear Diffusion.

My interest in the topic started decades ago under the influence of the late Philippe Bénilan who always thought about nonlinear diffusion problems in functional terms; his mind was busy with functional bounds and semigroups, and he made some of the basic contributions on which the text is built; in that connection and time, Laurent Véron had also a strong influence through his classical paper [Ve79]. Next come two of the main techniques: I learnt symmetrization from Giorgio Talenti and the art of self-similarity from Shoshana Kamin, Bert Peletier and Grisha Barenblatt. The books of the latter are a continuous source of inspiration and enjoyment and an open window into the Russian school of mathematics.
Many of the topics reported here originate from works with collaborators, too numerous to quote; I would like to single out the inspiration I received for this research from Don Aronson and Luis Caffarelli, with whom I spent happy periods in the USA and wrote some of my best contributions. Later, I was strongly influenced by Victor Galaktionov, who loves asymptotics. I would also like to thank Haim Brezis for his continuous encouragement of my mathematics; besides, he produced the first smoothing effects applicable to a large class of nonlinear evolution equations including the porous medium equation; he also pioneered the study of Radon measures as data, and he wrote with Avner Friedman a very influential paper on nonexistence for fast diffusion, a favorite topic for me. The presentation of the geometric aspects of fast diffusion owes much to conversations with Panagiota Daskalopolous. Work on the \( p \)-Laplacian was shared with Lucio Boccardo and Thierry Gallouët.

Finally, this work would not have been possible without the scientific contributions and personal help of my former students Ana Rodriguez, Arturo de Pablo, Fernando Quirós, Guillermo Reyes, Juan Ramón Esteban, Manuela Chaves, Omar Gil and Raúl Ferreira, to whom I would like to add Emmanuel Chasseigne and Matteo Bonforte.

The final index lists the main concepts and the names of the authors of the results that have been most influential on the author in writing this text, as mentioned in the different chapters. The author apologizes for undue omissions in the list and the citations.

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