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The definition of “definable orientation” in section 5 of [1] is not correct. Where it says: “for each point there is a definably compact neighbourhood (of the point) N and a class...”. It should say: “for each proper m-ball N of Y there is a class...”. (See section 4 in [1] for the definition of proper m-ball.) So the correct definition is:

**Definition.** A definable orientation of a definable manifold Y of dimension m is a map s which assigns to each point y ∈ Y a generator s(y) of the local definable homology group H_m^def(Y, Y - y) and which is locally constant in the following sense: for each proper m-ball B of Y there is a class ζ_B ∈ H_m^def(Y, Y - B) such that for each p ∈ B the natural homomorphism j_B^p : H_m^def(Y, Y - B) → H_m^def(Y, Y - p), induced by the inclusion map (Y, Y - B) → (Y, Y - p), sends ζ_B into s(p).

**Remark.** j_B^p is actually an isomorphism.

With this new definition the proof of Theorem 5.2 in [1] (the existence and unicity of a generator of H_m^def(X) compatible with a given orientation) should be changed accordingly as follows. As in [1], we prove the stronger result:

**Theorem.** If N is a definably compact subset of a definable manifold Y of dimension m with a definable orientation s, then there is one and only one class ζ_N ∈ H_m^def(Y, Y - N) such that for each p ∈ N, j_N^p maps ζ_N to s(p).

Proof. First observe that the proof of this statement, as it is in [1], proves the unicity of the relative homology class ζ_N. To prove the existence we use the unicity and we have to consider the following cases:

- Case (a). N is contained in a proper m-ball of Y. Then the existence of ζ_N is ensured by definition.
- Case (b). N = N_1 ∪ N_2 and there exist ζ_{N_1} and ζ_{N_2} both satisfying the above result. Then using a suitable Mayer-Vietoris sequence (as in case 2 of [1]) we can ensure the existence of the required ζ_N.
- Case (c): N is an arbitrary definably compact subset of Y. Then we argue as in case 5 of [1] to get first finitely many definably compact subsets N_1, ..., N_k of Y such that N = N_1 ∪ ⋯ ∪ N_k and each N_i is contained in a proper m-ball of Y, and then the result is obtained by induction on k using cases (a) and (b).

Note that an m-dimensional definable group G, equipped with its definable manifold structure, has a map s defined as in [1] (choose a generator s(x) ∈ H_m^def(G, G - x) at a point x of a given definably connected component of G and extend s to the whole component by left group multiplication). A routine verification shows that such a map s is a definable orientation according to the new definition (so the proof of Corollary 3.4 in [1] remains the same). To see this one uses the remark above and the fact that the composition (G, G - B) ↠ (G, G - x) ↠ (G, G - y) is definably homotopic to (G, G - B) ↠ (G, G - y), where i and j are inclusions and l_z is left multiplication by z.

Finally note that with the incorrect definition of orientation given in [1] leads to various pathologies such as the following. Let X be the unit circle in an o-minimal non-archimedean expansion of a field. Fix a point x_0 ∈ X in the circle and let I ⊆ X be the infinitesimal neighbourhood of x_0 ∈ X (a non-definable set). Then according to the definition of orientation given in [1] we could orient I in the clockwise direction and the complement of I in X in the opposite direction. Clearly it cannot exist a generator of H_1^def(X) compatible with this “orientation” of X.

References


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