Bloch waves homogenization of a Dirichlet problem in a periodically perforated domain

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Abstract

In this paper we use the spectral method of Bloch waves to study the homogenization process of the Poisson equation in a periodically perforated domain, under homogeneous Dirichlet conditions on both exterior and interior boundaries, as the hole size goes to zero more rapidly than the micro-structure size. Using this method, we find the exact value of the critical hole size, which separates the different behaviors, where the classical strange term may or may not appear in the homogenized equation. This strange term is related to the asymptotic behavior of the first eigenvalue with respect to the hole radius.

1 Introduction

The general question that forms the focus of this paper is the Bloch waves homogenization of Dirichlet’s type-like problem in perforated domains. For any \( \varepsilon > 0 \), we consider the set \( \Omega^\varepsilon \) obtained by removing from an open bounded set \( \Omega \subset \mathbb{R}^N \) (\( N = 2, 3 \)) a periodic network of balls in \( \mathbb{R}^N \). The periodicity of the medium is \( 2\pi \varepsilon \) and the radius of balls is \( r(\varepsilon) \). In this domain, we study the homogenization of the Poisson equation with homogeneous Dirichlet conditions on the boundary, including the boundaries of holes. In this paper, we are going to consider that the hole size \( r(\varepsilon) \) depends on the micro-structure size \( \varepsilon \) such that \( r(\varepsilon) \) goes to zero more rapidly than the micro-structure size.

As is well known, the homogenization process is concerned with obtaining suitable description of the asymptotic behavior of solutions of boundary value problems in such domains, as \( \varepsilon \) tends to zero. For a nice introduction to this subject, the reader is referred to the books of Bensoussan, Lions and Papanicolaou [3], Sanchez-Palencia [22]. Homogenization problems in a domain with small holes have been widely studied in the literature, using different methods, and we briefly mention just a few references: Marchenko and Khruslov [13, 14], Cioranescu and Saint Jean Paulin [5], Cioranescu and Murat [4], Oleinik and Shaposhnikova [17], Belyaev [2], Conca, Murat and Timofte [8] and the references therein.

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The method we used is the so-called Bloch waves method (see [3], [23], [11]). The Bloch waves are a family of eigenvalues and eigenvectors associated with partial differential equations and which provide the spectral resolution of elliptic differential operators. These waves transform the system of partial differential equations into a family of algebraic equations. The goal of this method is to show that the limit of this family, as the micro-structure size tends to zero, represents the homogenized equation written in the Fourier space.

In the sequel, we briefly recall some references about studying homogenization problems by using the Bloch waves decomposition. Conca and Vanninathan [11] used this method, in order to homogenized the classical problem of elliptic operators in arbitrary domains with periodically and symmetrically oscillating coefficients. Ganesh and Vanninathan [12] introduced the so-called dominant Bloch mode to prove the homogenization result of Murat and Tartar [16] in the general periodic case for non-selfadjoint operators. Conca, Orive and Vanninathan [9, 10] defined the Bloch approximation in order to obtain both the first and second order correctors of the homogenization problem.

In the case of periodically perforated domains, we can mention the works of Conca, Gómez, Lobo and Pérez [6, 7], where the authors have studied, by using Bloch waves, the asymptotic behavior of the solution of an elliptic boundary value problem with strongly oscillating coefficients and with a homogeneous Neumann condition on the holes boundary, in the case of a bounded periodically perforated domain where the structure periodicity and the hole size are of the same order as $\varepsilon$. Further, we mention the work of Ortega, San Martín and Smaranda [18] where it has been studied, by using the spectral method of Bloch waves, the homogenization of the Laplace equation in a periodically perforated domain, under a non-homogeneous Neumann condition on the boundary of the holes, as the size of the holes goes to zero more rapidly than the domain period.

The main novelty of this paper is that by using the Bloch waves decomposition technique, we can characterize the limit of the solution of previous problem, as $\varepsilon$ goes to zero. We recover the different behaviors, depending on the way as the radius $r(\varepsilon)$ goes to zero, which are completely characterized by the technique we use. In the particular case, where $r(\varepsilon)$ converges to zero in a critical way, on the homogenized equation appears the so called "strange term", which is a consequence of the asymptotic behavior of the first eigenvalue of a suitable spectral problem.

The homogenization result presented here is not new, however, the presentation is original and, it is hoped, clear and easy to understand. The same problem treated in this paper by using the Bloch waves, has been studied by Cioranescu and Murat in [4] who employed Tartar's test functions method (see Murat and Tartar [16]). Further, by using the spectral method, we show the connection between "the strange term coming from nowhere" and the asymptotic behavior with respect to the hole size of the first eigenvalue associated with the problem. This asymptotic behavior, stated in Theorem 3.3, is a generalization of Ozawa's work [19]. In [19] the author deals with the asymptotic behavior of the eigenvalues associated with the Laplace operator in the particular case of a bounded domain with an unique hole and with Dirichlet's type conditions on both boundaries. Following the same approach here, we find a similar asymptotic behavior of the first eigenvalue with respect to the hole size. Using this last result, we also determine the exact value of the critical hole size for which the strange term appears in the homogenized equation.

The outline of the remaining paper is the following. The next section is devoted to the setting of the problem and statement of the main homogenization result. Section 3, divided into three subsections, deals with the asymptotic behavior of the spectrum of the Laplace operator in $\mathbb{R}^N$ periodically perforated by balls, as the balls radius goes to zero. In Section 4, we introduce the Bloch waves associated with our problem, we study their regularity properties and we prove some
preliminary results necessary for the proof of the main homogenization result which is given in Section 5.

2 Setting of the problem and statement of main results

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be an open bounded set with a smooth enough boundary. For any positive $\varepsilon$, let us perforate the set $\Omega$ by a periodic network of balls of radius $r(\varepsilon)$ and centered in $2\pi\varepsilon\mathbb{Z}^N$, denoted by $T^\varepsilon$, that is,

$$T^\varepsilon = \bigcup_{p \in \mathbb{Z}^N} \mathcal{B}(2\pi p\varepsilon, r(\varepsilon)),$$

where we assume that $r : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ is a continuous map satisfying the condition $r(\varepsilon) < \pi \varepsilon$.

This periodically perforated domain will be denoted by $\Omega^\varepsilon$, more precisely,

$$\Omega^\varepsilon = \Omega \setminus T^\varepsilon.$$

Let us notice that all holes have the same shape (are balls) and since the distance between two adjacent perforations is $2\pi\varepsilon$, the condition $r(\varepsilon) < \pi \varepsilon$ is necessary in order to prevent the overlapping of the holes.

We are interested in studying the asymptotic behavior, as $\varepsilon$ goes to zero, of the solution $u^\varepsilon$ of the following homogeneous Dirichlet boundary-value problem:

$$\begin{cases} -\Delta u^\varepsilon = f & \text{in } \Omega^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial \Omega^\varepsilon, \end{cases} \quad (2.1)$$

where $f$ is a given function in $L^2(\Omega)$. This asymptotic behavior will be studied when the hole radius goes to zero more rapidly than the micro-structure size, i.e.,

$$\lim_{\varepsilon \to 0} \frac{r(\varepsilon)}{\varepsilon} = 0. \quad (2.2)$$

The asymptotic behavior of the solution $u^\varepsilon$ has already studied in Cioranescu and Murat [4] by using the Tartar’s test functions method. Using this technique, the authors proved that if the hole radius is equal to a critical size, in the homogenized equation appears the so-called strange term. The main novelty of our work is that using the Bloch waves spectral method, we not only can recover the previous result, but also we find the exact value of the critical size and the strange term as direct consequences of this homogenization technique. In particular, we show that "the strange term coming from nowhere" is related to the asymptotic behavior of the first eigenvalue associated with the problem.

Let us now state the homogenization result that we will prove in this paper, by using the Bloch waves decomposition, result concerning with the asymptotic behavior of solution $u^\varepsilon$ when $\varepsilon \to 0$. Depending on the way as $\frac{r(\varepsilon)}{\varepsilon}$ goes to zero, we obtain different asymptotic behaviors of the solution $u^\varepsilon$. These different asymptotic behaviors depend on the ratio:

$$\frac{R(\frac{r(\varepsilon)}{\varepsilon})}{\varepsilon^2},$$

where $R$ is a real function, dependent on the dimension of the space, which is given by the following formula:

$$R(x) = \begin{cases} - (\ln x)^{-1} & \text{if } N = 2, \\ x & \text{if } N = 3. \end{cases} \quad (2.3)$$

The main homogenization result is the following:
Theorem 2.1. Let $u^\varepsilon \in H^1_0(\Omega^\varepsilon)$ be the sequence of unique solutions of the problem (2.1). Then, the extensions by zero inside of the holes of the solutions $u^\varepsilon$, denoted by $\tilde{u}^\varepsilon$, satisfy
\[ \tilde{u}^\varepsilon \rightharpoonup u \quad \text{weakly in } H^1_0(\Omega). \]

Depending on the hole radius $r(\varepsilon)$, we have the following characterizations of the limit $u$:

(i) If $\lim_{\varepsilon \to 0} \varepsilon^{-2} R(\frac{r(\varepsilon)}{\varepsilon}) = \ell$, where $\ell > 0$, then the function $u$ is the unique solution of the following homogenized problem:
\[
\begin{aligned}
(S_N \phi_1^2(0) \ell) u - \Delta u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
where $S_N$ denotes the area of the unit sphere in $\mathbb{R}^N$ and $\phi_1$ is the first normalized eigenvector of the periodic Laplace operator in $Y = [-\pi, \pi]^N$.

(ii) If $\lim_{\varepsilon \to 0} \varepsilon^{-2} R(\frac{r(\varepsilon)}{\varepsilon}) = 0$, then the limit $u$ is the unique solution of the problem
\[
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(iii) If $\lim_{\varepsilon \to 0} \varepsilon^{-2} R(\frac{r(\varepsilon)}{\varepsilon}) = +\infty$, then $u$ is identically equal to zero.

One of the key ingredients in the proof of the above Theorem, by using the spectral method of Bloch waves, consists in the asymptotic behavior of the first eigenvalue of the Laplace operator in $\mathbb{R}^N$ periodically perforated with balls, as their radius go to zero. This asymptotic behavior will be developed in next section and stated in Theorem 3.3.

3 Asymptotic behavior with respect to the hole radius

In this section, we are going to study the asymptotic behavior of the spectrum of the Laplace operator in $\mathbb{R}^N$ periodically perforated with balls of radius $a$, as $a$ goes to zero. We begin by introducing some notations and stating two theorems. The Theorem 3.1 states the convergence of the spectrum of Laplace operator in a periodically perforated domain to the corresponding spectrum of the same operator in a periodic domain without holes. The second result stated in Theorem 3.3 gives the rate of previous convergence, but only for simple eigenvalues.

3.1 Statement of results on the asymptotic behavior

In order to state this asymptotic behavior let us begin by introducing the following notations.

For a given $a$, small enough, let denote by $S^a$ the $2\pi$-periodic network of holes of radius $a$ in $\mathbb{R}^N$, that is,
\[ S^a = \bigcup_{p \in \mathbb{Z}^N} \mathcal{B}(2\pi p, a). \]

Additionally, we consider the reference cell $Y = [-\pi, \pi]^N$ in $\mathbb{R}^N$ and the corresponding perforated reference cell
\[ Y^*_a = Y \setminus \mathcal{B}(0, a). \]
Let us introduce the following functional spaces that will be used in the remaining paper:

\[
L^2_\#(Y) = \{ \phi \in L^2_{\text{loc}}(\mathbb{R}^N) \mid \phi \text{ is } Y\text{-periodic} \},
\]

\[
H^1_\#(Y) = \{ \phi \in L^2_\#(Y) \mid \nabla \phi \in L^2_\#(Y)^N \},
\]

\[
L^2_\#(Y^*) = \{ \phi \in L^2_{\text{loc}}(\mathbb{R}^N \setminus \overline{S^a}) \mid \phi \text{ is } Y\text{-periodic} \},
\]

\[
H^1_\#(Y^*) = \{ \phi \in L^2_\#(Y^*) \mid \nabla \phi \in L^2_\#(Y^*)^N, \ \phi = 0 \text{ on } \partial S^a \}.
\]

Let us consider the following spectral problem in \( \mathbb{R}^N \) perforated by the network \( S^a \): find \( \lambda(a) \in \mathbb{R} \) and \( \phi(a; \cdot) \neq 0 \) such that

\[
\begin{cases}
-\Delta \phi(a; \cdot) = \lambda(a) \phi(a; \cdot) \quad \text{in } \mathbb{R}^N \setminus \overline{S^a}, \\
\phi(a; \cdot) = 0 \quad \text{on } \partial S^a,
\end{cases}
\] (3.1)

\( \phi(a; \cdot) \) is \( Y \)-periodic.

It is well known that the above spectral problem admits a countable sequence of strictly positive eigenvalues, each of them having finite multiplicity. As usual, we arrange them in increasing order repeating each eigenvalue according to its multiplicity:

\[0 < \lambda_1(a) \leq \lambda_2(a) \leq \ldots \leq \lambda_m(a) \leq \ldots \to \infty.\]

We shall prove that, as \( a \) goes to zero, each eigenvalue converges to the corresponding eigenvalue of the Laplace operator in \( \mathbb{R}^N \) without holes. For this reason, let us introduce this limit spectral problem: find \( \lambda \in \mathbb{R} \) and \( \phi(\cdot) \neq 0 \) such that

\[
\begin{cases}
-\Delta \phi = \lambda \phi \quad \text{in } \mathbb{R}^N, \\
\phi \text{ is } Y\text{-periodic},
\end{cases}
\] (3.2)

which has also a countable sequence of positive eigenvalues, each of them having finite multiplicity:

\[0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_m \leq \ldots \to \infty.\]

Using the above notations we are able to state the two principal theorems of this section.

**Theorem 3.1.** For any \( m \geq 1 \), the \( m \text{th} \) eigenvalue of the spectral problem (3.1) converges to the corresponding \( m \text{th} \) eigenvalue of the limit spectral problem (3.2), i.e.,

\[\lambda_m(a) \longrightarrow \lambda_m \quad \text{as } a \to 0.\]

Using the simplicity of the first eigenvalue \( \lambda_1 \) we get the following corollary:

**Corollary 3.2.** For any \( a \) small enough, the first eigenvalue of (3.1) is simple.

In addition, we observe that \( \lambda_1(a) \longrightarrow 0 \), as \( a \) goes to zero. Let us now go further and in the following theorem we give the rate of the above convergence, which depends on the dimension of the space.

**Theorem 3.3.** As \( a \) goes to zero, the asymptotic behavior of the first eigenvalue of the spectral problem (3.1) is the following:

\[\lambda_1(a) = -2\pi (\ln a)^{-1} \phi_1^2(0) + O\left( (\ln a)^{-2} \right) \quad \text{if } N = 2,\]

\[\lambda_1(a) = 4\pi a \phi_1^2(0) + O\left( a^{3/2} \right) \quad \text{if } N = 3,\]

where \( \phi_1 \) is the first normalized eigenvector of (3.2).
Remark 3.4. As \( a \) goes to zero, one can prove that the same asymptotic behavior holds for all simple eigenvalues of the spectral problem (3.1).

Remark 3.5. Using the real function \( R \) defined in (2.3), one can write the following asymptotic behavior:

\[
\lambda_1(a) = S_N \phi_1^2(0) R(a) + O(R(a)^{3/2}),
\]

where in the two dimensional case, the error’s order is weaker than the one given by Theorem 3.3. However, for the proof of the homogenization result stated in Theorem 2.1, this last asymptotic behavior is enough.

A similar result, in the case of a bounded domain with an unique perforation, has been obtained by Ozawa [19] for the asymptotic behavior of the eigenvalues associated with the Laplace operator under Dirichlet type boundary condition. Also, Mazja, Nazarov and Plamenevskij present a general and unified approach to the asymptotic analysis of elliptic boundary value problems in singularly perturbed domains in [15].

### 3.2 Proof of Theorem 3.1

First, we proceed by proving the following density Lemma:

**Lemma 3.6.** Let us consider the following space

\[
V = \{ \varphi \in H^1_\#(Y) \mid \exists \{\varphi^a\}_{a \geq 0} \text{ such that } \varphi^a \in H^1_{0,\#}(Y^*_a) \text{ and } E^a \varphi^a \rightharpoonup \varphi \text{ strong in } H^1_\#(Y) \},
\]

where \( E^a \) denotes the extension by zero inside of the hole \( \mathcal{B}(0,a) \). Then \( V \) is dense in the space \( H^1_\#(Y) \).

**Proof.** First, we define the space

\[
W = \{ \rho \in H^1_\#(Y) \mid \text{supp}(\rho) \subset \subset \overline{Y} \setminus \{0\} \}.
\]

By convolution technique, we deduce that the space of \( C^\infty \)-functions with compact support in \( Y \setminus \{0\} \) and \( Y \)-periodicity conditions, denoted by \( \mathcal{D}_\#(Y \setminus \{0\}) \), is dense in the space \( W \). On the other hand, by regularization method, one can prove the density of \( W \) in \( H^1_\#(Y) \), for any \( N \geq 2 \). Since \( \mathcal{D}_\#(Y \setminus \{0\}) \subset V \subset H^1_\#(Y) \), we conclude.

**Proof of Theorem 3.1.** Let us consider the Green operator

\[
H^a : L^2_\#(Y^*_a) \rightarrow L^2_\#(Y^*_a)
\]

\[
f \mapsto \varphi^a = H^a f,
\]

where \( \varphi^a \) is the unique solution of the following variational problem

\[
\begin{cases}
\text{Find } \varphi^a \in H^1_{0,\#}(Y^*_a) \text{ such that } \\
B^a(\varphi^a, \varphi_1) = \int_{Y^*_a} f \varphi_1 \quad \forall \varphi_1 \in H^1_{0,\#}(Y^*_a),
\end{cases}
\]

with the bilinear form \( B^a(\cdot, \cdot) \) defined as follows:

\[
B^a(\varphi^a, \varphi_1) = \int_{Y^*_a} \varphi^a \varphi_1 + \int_{Y^*_a} \nabla \varphi^a \cdot \nabla \varphi_1.
\]
Now, we consider the limit operator

$$H : L^2_\#(Y) \rightarrow L^2_\#(Y)$$

$$g \mapsto \varphi = Hg,$$

where \(\varphi\) is the unique solution of the variational problem

$$\begin{cases}
\text{Find } \varphi \in H^1_\#(Y) \text{ such that } \\
B(\varphi, \varphi_1) = \int_Y g \overline{\varphi_1} \quad \forall \varphi_1 \in H^1_\#(Y),
\end{cases} \quad (3.4)$$

with

$$B(\varphi, \varphi_1) = \int_Y \varphi \overline{\varphi_1} + \int_Y \nabla \varphi \cdot \nabla \overline{\varphi_1}.$$

We can remark that \(H^a\) and \(H\) are compact self-adjoint operators in \(\mathcal{L}(L^2_\#(Y^*_a))\), respectively in \(\mathcal{L}(L^2_\#(Y))\). Their spectrum consists in countable sequence of eigenvalues converging to zero.

We define an extended operator \(\widehat{H}^a \in \mathcal{L}(L^2_\#(Y))\) as follows:

$$\widehat{H}^a = E^a H^a P^a,$$

where \(P^a\) and \(E^a\) are respectively a projection from \(L^2_\#(Y)\) into \(L^2_\#(Y^*_a)\) and an extension by zero from \(L^2_\#(Y^*_a)\) into \(L^2_\#(Y)\). We can prove that the operators \(P^a\) and \(E^a\) are adjoint one from the other, which implies that the operator \(\widehat{H}^a\) is still self-adjoint. Moreover, since the product \(P^a E^a\) is equal to the identity in \(L^2_\#(Y^*_a)\), we conclude that \(H^a\) and \(\widehat{H}^a\) have the same spectrum.

In the sequel, we prove that the operator \(\widehat{H}^a\) uniformly converges to \(H\) in the space \(\mathcal{L}(L^2_\#(Y))\). For this purpose, let consider an arbitrary function \(g \in L^2_\#(Y)\). Due to the variational formulation (3.3) with \(f = P^a g\) and the particular test function \(H^a P^a g \in H^1_\#(Y^*_a)\), we get

$$\|\widehat{H}^a g\|_{H^1_\#(Y)} \leq \|g\|_{L^2_\#(Y)}. \quad (3.5)$$

Therefore, there exists a subsequence, still denoted by \(\widehat{H}^a g\), and a function \(\chi \in H^1_\#(Y)\) such that

$$\widehat{H}^a g \rightharpoonup \chi \quad \text{weakly in } H^1_\#(Y). \quad (3.6)$$

Now, we shall identify \(\chi\) as the solution of the problem (3.4). For this purpose, let consider \(\varphi_1 \in V\). By definition of \(V\), there exists a sequence \(\{\varphi^a_1\}_{a \geq 0}\) such that \(\varphi^a_1 \in H^1_{0,\#}(Y^*_a)\) and \(E^a \varphi^a_1 \rightharpoonup \varphi_1\) strongly in \(H^1_\#(Y)\). Using this sequence as test function in the variational formulation (3.3), we have that

$$B^a(H^a P^a g, \varphi^a_1) = \int_Y g \overline{E^a \varphi^a_1} \rightarrow \int_Y g \overline{\varphi_1}.$$

On the other hand, from the definition of the bilinear form \(B^a\), we have

$$B^a(H^a P^a g, \varphi^a_1) = \int_Y \widehat{H}^a g \overline{E^a \varphi^a_1} + \int_Y \nabla \widehat{H}^a g \cdot \nabla \overline{E^a \varphi^a_1},$$

where, passing to the limit as \(a\) goes to zero and using the convergence (3.6), we get

$$B^a(H^a P^a g, \varphi^a_1) \rightarrow \int_Y \chi \overline{\varphi_1} + \int_Y \nabla \chi \cdot \overline{\varphi_1} = B(\chi, \varphi_1).$$
Hence, due to the uniqueness of the limit and Lemma 3.6, we deduce that

$$B(\chi, \varphi_1) = \int_Y g \varphi_1 \quad \forall \varphi_1 \in H^1_\#, \tag{3.7}$$

which implies that $\chi = Hg$.

Using Theorem 2.2 from Conca and Allaire [1], it follows that the eigenvalues of the operator $H^a$ converge to the corresponding eigenvalues of the operator $H$. This completes the proof. \( \square \)

### 3.3 Proof of Theorem 3.3

We shall give in details the proof of Theorem 3.3 in 3-dimensional case. In the 2-dimensional case, the proof is completely analogous, but since most of the formulae involved have different analytic expressions, we shall omit to treat it.

Let us start the proof by studying some useful properties of the Green function associated with the Laplace plus identity operator in the domain $Y$ with $Y$-periodicity conditions. More precisely, let us consider the Green function $G(x, y)$, solution of the following problem:

$$(I - \Delta_x) G(x, y) = \delta(x - y) \quad x, y \in Y,$n

$G(\cdot, y)$ is $Y$-periodic, $y \in Y$,

where $I$ is the identity operator, and $\delta$ denotes the Dirac function.

We observe that the operator $-\Delta$ is not injective in $L^2_\#(Y)$, because its first eigenvalue is equal to zero. For this reason, it is necessary to study the operator $I - \Delta$, whose eigenvectors are the same as the ones of the operator $-\Delta$, and the corresponding eigenvalues are strictly positive.

In Stein and Weiss [24, Teorema 2.17] has been studied the Green function in the particular case $y = 0$, and it was found the following asymptotic behavior, as $x \to 0$:

$$G(x, 0) = \frac{1}{4\pi|x|} + \mathcal{O}(1). \tag{3.8}$$

In order to obtain the Green function in the general case ($y \neq 0$), because of the periodicity property, it is enough to translate the previous function, i.e.,

$$G(x, y) = G(x - y, 0).$$

Let us define the number $K(a) = (4\pi)^{-1}a^{-1}$ and let consider the sets $\omega_a$ and $\beta_a$ defined by

$$\omega_a = \{x \in Y \mid G(x, 0) \leq K(a)\},$$

$$\beta_a = Y \setminus \overline{\omega_a}.$$

Due to the asymptotic behavior (3.8), for $a$ small enough, we observe that the set $\beta_a$ is a neighborhood of the origin, where the Green function goes to infinity. Then, the set $\omega_a$ corresponds to the domain where the Green function is bounded. In general, the domain $\beta_a$ is not a ball centered in the origin. In the following Lemma, we can see how this domain is bounded by two balls of radius of the same order as $a$.

**Lemma 3.7.** There exists a positive constant $C$, independent of $a$, such that

$$\omega_{a + Ca^2} \subset Y^* \subset \omega_{a - Ca^2}.$$
Proof. The result is a direct consequence from the definition of the domain $\omega_a$ and the asymptotic behavior around zero of the Green function $G(x,0)$.

Remark 3.8. We observe that the inclusion
\[ \omega_{2a} \subset Y^*_a \subset \omega_{a/2}, \]  
holds, but it is a particular case of the previous one states in Lemma 3.7.

Let us state the following Lemma that will be useful in the proof of Theorem 3.3.

Lemma 3.9. If $u_a$ is a $Y$-periodic function satisfying $(I - \Delta) u_a = 0$ in $\omega_a$ and $|u_a| \leq 1$ on $\partial \beta_a$, then there exists a constant $C > 0$, independent of $a$, such that
\[ |u_a(x)| \leq C a G(x,0) \quad \forall x \in \omega_a. \]

Proof. We consider the function $\Phi_a(x) = 4\pi a G(x,0)$ that satisfies $(I - \Delta) \Phi_a = 0$ in $\omega_a$ and $\Phi_a = 1$ on $\partial \beta_a$. Therefore, by the maximum principle we get that $\Phi_a$ is a $Y$-periodic bound of $u_a$.

Let $G_a(x,y)$ be the Green function associated with the Laplace plus identity operator, $Y$-periodic in $\omega_a$ and with Dirichlet boundary conditions on $\partial \beta_a$. Its asymptotic behavior, as $a \to 0$, is given by:

Theorem 3.10. For any fixed $x, y \in Y \setminus \{0\}$, $x \neq y$, the asymptotic relation
\[ G_a(x,y) = G(x,y) - K(a)^{-1}G(x,0)G(y,0) + \mathcal{O}(a^2) \]
holds as $a \to 0$.

Remark 3.11. It should be remarked that the term $\mathcal{O}(a^2)$ is not uniform with respect to $x$ and $y$.

Proof. We consider $x, y \in Y \setminus \{0\}$. We denote
\[ g_a(x,y) = G_a(x,y) - G(x,y) + K(a)^{-1}G(x,0)G(y,0). \]
As a function dependent on $x$, $g_a(x,y)$ has the following properties:
\[ (I - \Delta_x)g_a(x,y) = 0 \quad \text{in } \omega_a \quad \text{and} \quad g_a(\cdot,y) \text{ is } Y\text{-periodic}. \]
Due to the definition of the domain $\beta_a$, we have that on the boundary $\partial \beta_a$,
\[ g_a(x,y)|_{x \in \partial \beta_a} = G(y,0) - G(y,x). \]
Therefore, by using the regularity of the Green function $G(y,\cdot)$, as $a$ small enough, we deduce the existence of a positive constant $C$, independent of $a$, such that
\[ |g_a(x,y)|_{x \in \partial \beta_a} \leq Ca. \]
Let us remark that $C$ depends on the distance between $y$ and the origin.

Due to Lemma 3.9 applied for the function $g_a(x,y)/(Ca)$, we conclude.
Now, let us consider the bounded linear operator $G_a$ in $L^2_{\#}(\omega_a)$ associated with the Green function $G_a(x,y)$, that is,

$$(G_a f)(x) = \int_{\omega_a} G_a(x,y) f(y) \, dy.$$  

In order to approximate this operator, we define

$$h_a(x,y) = G(x,y) - K(a)^{-1} G(x,0) G(y,0),$$

and

$$(H_a f)(x) = \int_{\omega_a} h_a(x,y) f(y) \, dy.$$  

We observe that $H_a$ is a bounded linear operator in $L^2_{\#}(\omega_a)$. To study the difference between these two operators, we denote by $Q_a = H_a - G_a$, and we prove the following Lemma:

**Lemma 3.12.** For any $f \in L^2_{\#}(\omega_a)$, the function $Q_a f$ is $Y$-periodic and satisfies $(I - \Delta)Q_a f = 0$ in $\omega_a$ and

$$\sup_{x \in \partial \beta_a} |(Q_a f)(x)| \leq I(a) \|f\|_{L^2_{\#}(\omega_a)},$$

where

$$I(a) = \sup_{x \in \partial \beta_a} \left( \int_{\omega_a} [G(x,y) - G(0,y)]^2 \, dy \right)^{1/2} \leq C a^{1/2}.$$  

**Proof.** It is clear that $Q_a f$ is $Y$-periodic and satisfies

$$(I - \Delta)Q_a f = 0 \quad \text{in} \; \omega_a.$$  

Moreover, since $G(x,0) = K(a)$ and $G_a(x,y) = 0$ on $\partial \beta_a$, we get

$$(Q_a f)(x) = \int_{\omega_a} [G(x,y) - G(0,y)] f(y) \, dy \quad \forall x \in \partial \beta_a,$$

and therefore, the relation (3.11) is a direct consequence of the Cauchy-Schwarz inequality.

Let us now estimate the term $I(a)$. From its definition, since the integral in the variable $y$ is extended until the boundary $\partial \beta_a$, we add and subtract the terms $(4\pi)^{-1}|y|^{-1}$ and $(4\pi)^{-1}|x-y|^{-1}$. Then, $I(a)$ can be bounded by the sum of the following two terms:

$$X_1(a) = \sup_{x \in \partial \beta_a} \left( \int_{\omega_a} \left[ \left( G(x,y) - \frac{1}{4\pi|x-y|} \right) - \left( G(0,y) - \frac{1}{4\pi|y|} \right) \right]^2 \, dy \right)^{1/2},$$

and

$$X_2(a) = \sup_{x \in \partial \beta_a} \left( \int_{\omega_a} \left[ \frac{1}{4\pi|x-y|} - \frac{1}{4\pi|y|} \right]^2 \, dy \right)^{1/2}.$$  

In order to estimate the integral $X_1(a)$, let us observe that the Green function minus its singularity is a $C^\infty$-class function, and therefore we have that

$$X_1(a) = O(a).$$
Let us now estimate the integral $X_2(a)$ and we have

$$X_2(a)^2 = \left(\frac{1}{4\pi}\right)^2 \sup_{x \in \partial \beta_a} \int_{\omega_a} \left[\frac{1}{|x-y|} - \frac{1}{|y|}\right]^2 \, dy$$

$$\leq \left(\frac{1}{4\pi}\right)^2 \sup_{x \in \partial \beta_a} |x| \int_{\mathbb{R}^3} \left[\frac{1}{|x/|x| - z|} - \frac{1}{|z|}\right]^2 \, dz$$

$$\leq C \sup_{x \in \partial \beta_a} |x|.$$ 

Since $|x| \leq 2a$, it follows that $X_2(a)^2 \leq C \, a$. 

As a consequence of Lemma 3.9 and Lemma 3.12, we get the following Proposition:

**Proposition 3.13.** For any a small enough, there exists $C > 0$, independent of $a$, such that

$$\|H_a - G_a\|_{L(L_2^2(\omega_a))} \leq Ca^{3/2}.$$ 

**Proof.** Let $f \in L_2^2(\omega_a)$ be an arbitrary function with norm equal to 1. By using Lemma 3.12, we get that the function $Q_a f$ is $Y$-periodic, it satisfies $(I - \Delta)Q_a f = 0$ in $\omega_a$, and on the boundary $\partial \beta_a$ is bounded by $Ca^{1/2} \|f\|_{L_2^2(\omega_a)}$. Then, since $\|f\|_{L_2^2(\omega_a)} = 1$, we use Lemma 3.9 to deduce that

$$|(Q_a f)(x)| \leq Ca^{3/2} G(x, 0) \quad \forall x \in \omega_a.$$ 

Hence, taking the $L_2^2(\omega_a)$-norm, we conclude the result. 

Let us now introduce the operators associated with the Green functions $G(x, y)$ and $h_a(x, y)$ defined in (3.10), in the following way:

$$(Gf)(x) = \int_Y G(x, y) f(y) \, dy$$

and

$$(\tilde{H}_a f)(x) = \int_Y h_a(x, y) f(y) \, dy. \tag{3.12}$$

We observe that these operators are linear and bounded in the space $L_2^2(Y)$.

In the sequel, let us denote by the Greek letter $\mu$ the eigenvalues of the operator $G$. Since the first eigenvalue of the Laplace plus identity operator in $Y$ with $Y$-periodicity conditions, denoted by $\nu_1 = 1$, is simple, then $\mu_1 = 1 / \nu_1 = 1$ is a simple eigenvalue of the operator $G$. Therefore, the problem

$$(G - \mu_1)\xi(x) = \mu_1 \phi_1(0) [G(x, 0) - \mu_1 \phi_1(0) \phi_1(x)], \tag{3.13}$$

$$\int_Y \xi(x) \phi_1(x) \, dx = 0, \tag{3.14}$$

admits an unique solution in $L_2^2(Y)$, due to the fact that the right hand side of the equation (3.13) is orthogonal to the eigenvector $\phi_1$ in $L_2^2(Y)$. 

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By using this function $\xi$, we are going to construct approximations of the eigenvalues and eigenvectors associated with the operators $\tilde{H}_a, H_a$ and $G_a$. We introduce

$$\tilde{\mu}_1(a) = \mu_1 - K(a)^{-1}\mu_1^2\phi_1^2(0), \tag{3.15}$$

$$\tilde{\phi}_1(a) = \phi_1 + K(a)^{-1}\xi. \tag{3.16}$$

In the following Lemma, we precise in what sense the previous quantities are approximations of the eigenvalues and eigenvectors associated with the operator $\tilde{H}_a$.

**Lemma 3.14.** There exists a positive constant $C$, independent of $a$, such that

$$\| (\tilde{H}_a - \tilde{\mu}_1(a))\tilde{\phi}_1(a) \|_{L^2(Y)} \leq C K(a)^{-2}.$$

**Proof.** Due to the definition of the operator $\tilde{H}_a$, (see (3.10) and (3.12)) we have that

$$(\tilde{H}_a f)(x) = (Gf)(x) - K(a)^{-1}G(x, 0)(Gf)(0).$$

Using this expression in the particular cases $f = \xi$ and $f = \phi_1$, we get that

$$(\tilde{H}_a \xi)(x) = \mu_1 \xi(x) + \mu_1 \phi_1(0) [G(x, 0) - \mu_1 \phi_1(0) \phi_1(x)] - K(a)^{-1}G(x, 0)(G\xi)(0),$$

respectively

$$(\tilde{H}_a \phi_1)(x) = \mu_1 \phi_1(x) - K(a)^{-1}G(x, 0)\mu_1 \phi_1(0).$$

Combining these previous identities with (3.16), we deduce

$$(\tilde{H}_a \tilde{\phi}_1(a))(x) = \mu_1 \tilde{\phi}_1(a)(x) - \mu_1^2 \phi_1^2(0)K(a)^{-1}\tilde{\phi}_1(a)(x)$$

$$+ \mu_1^2 \phi_1^2(0)K(a)^{-2}\xi(x) - K(a)^{-2}G(x, 0)(G\xi)(0),$$

then, due to (3.15), we get

$$(\tilde{H}_a - \tilde{\mu}_1(a))\tilde{\phi}_1(a)(x) = K(a)^{-2} (\mu_1^2 \phi_1^2(0)\xi(x) - G(x, 0)(G\xi)(0)).$$

Thus, we get the result by taking the $L^2(Y)$-norm. \hfill \square

In the following Lemma we are looking for a relation between the operators $\tilde{H}_a$ and $H_a$.

**Lemma 3.15.** There exists a positive constant $C$, independent of $a$, such that

$$\| \tilde{H}_a \tilde{\phi}_1(a) - H_a \chi_{\omega_a} \tilde{\phi}_1(a) \|_{L^2(Y)} \leq C a^{3/2},$$

where $\chi_{\omega_a}$ is the characteristic function on the domain $\omega_a$.

**Proof.** Using definitions of the operators $\tilde{H}_a$ and $H_a$, we deduce that

$$\left[ \tilde{H}_a \tilde{\phi}_1(a) - H_a \chi_{\omega_a} \tilde{\phi}_1(a) \right](x) = \int_{\beta_a} h_a(x, y) \tilde{\phi}_1(a)(x) \ dy.$$

We denote by $f_a(x)$ the previous function, that satisfies $(I - \Delta)f_a = 0$ in $\omega_a$ and that is $Y$-periodic. Moreover, using the Cauchy-Schwarz inequality, we get that on the boundary $\partial \beta_a$, the function $f_a$ is uniformly bounded as follows:

$$|f_a(x)| \leq J(a) \| \tilde{\phi}_1(a) \|_{L^2(Y)} \quad \forall x \in \partial \beta_a,$$
where

\[ J(a) = \sup_{x \in \partial \beta_a} \left( \int_{\beta_a} h_a(x, y)^2 \, dy \right)^{1/2}. \]

Let us now estimate the term \( J(a) \). We observe that for all \( x \in \partial \beta_a \), we have \( G(x, 0) = K(a) \), and thus the function \( h_a(x, y) \) is equal to \( G(x, y) - G(0, y) \). Therefore, the integral \( J(a) \) can be bounded as follows:

\[ J(a) \leq \sup_{x \in \partial \beta_a} \left( \int_{\beta_a} G(x, y)^2 \, dy \right)^{1/2} + \sup_{x \in \partial \beta_a} \left( \int_{\beta_a} G(y, 0)^2 \, dy \right)^{1/2}. \]

We estimate the previous two integrals by using the asymptotic behavior (3.8) of the Green function. Thus, we get that there exists a positive constant \( C \), independent of \( a \), such that \( J(a) \leq C a^{1/2} \).

Due to Lemma 3.9 applied for the function \( f_a(x)/(C a^{1/2} \| \tilde{\phi}_1(a) \|_{L^2_\#(Y)}) \), we deduce that

\[ |f_a(x)| \leq C a^{3/2} \| \tilde{\phi}_1(a) \|_{L^2_\#(Y)} G(x, 0) \quad \forall x \in \omega_a. \]

Taking the \( L^2_\#(\omega_a) \)-norm, we conclude. \( \square \)

As a consequence of Lemma 3.14 and Lemma 3.15, we have the following inequalities:

\[
\| (G_a - \tilde{\mu}_1(a)) \chi_{\omega_a} \tilde{\phi}_1(a) \|_{L^2_\#(Y)} \leq \| H_a - G_a \|_{L(1_{\omega_a})} \| \tilde{\phi}_1(a) \|_{L^2_\#(\omega_a)} + \| \tilde{H}_a \tilde{\phi}_1(a) - H_a \chi_{\omega_a} \tilde{\phi}_1(a) \|_{L^2_\#(\omega_a)} \\
+ \| (\tilde{H}_a - \tilde{\mu}_1(a)) \tilde{\phi}_1(a) \|_{L^2_\#(Y)} \leq C a^{3/2}.
\]

This last inequality shows us that \( \tilde{\mu}_1(a) \) and \( \chi_{\omega_a} \tilde{\phi}_1(a) \) are approximations of one eigenvalue, respectively of one eigenvector associated with the operator \( G_a \). Moreover, this inequality allows us to deduce the existence of one eigenvalue of the operator \( G_a \) in a neighborhood of \( \tilde{\mu}_1(a) \). For this purpose, we use the following Lemma, that is a simplified version of Lemma 2 from Ozawa [19]:

**Lemma 3.16.** Let \( X \) be a Hilbert real space with the norm \( \| \cdot \| \) and let consider \( A \) a compact self-adjoint operator in \( X \). Assume that \( \lambda \in \mathbb{R} \setminus \{0\} \) and \( v \in X \) satisfy the properties \( \| (A - \lambda) v \| \leq \delta \) (with \( \delta > 0 \)) and \( \| v \| = 1 \), then there exists at least one eigenvalue \( \lambda^* \) of the operator \( A \) that satisfies the following inequality:

\[ |\lambda^* - \lambda| \leq 2\delta. \]

Applying the previous Lemma to the operator \( G_a \), it follows the existence of one eigenvalue of \( G_a \), denoted by \( \tilde{\mu}^*(a) \), such that

\[ |\tilde{\mu}_1(a) - \tilde{\mu}^*(a)| \leq C a^{3/2}. \]

This last inequality and the definition (3.15) of \( \tilde{\mu}_1(a) \) show us that

\[ \mu_1 - K(a)^{-1} \mu_1^2 \phi_1^2(0) - O(a^{3/2}) \leq \tilde{\mu}^*(a) \leq \mu_1 - K(a)^{-1} \mu_1^2 \phi_1^2(0) + O(a^{3/2}). \quad (3.17) \]

In order to conclude the proof of Theorem 3.3 it is necessary to consider a similar operator as \( G_a \), but defined in the domain \( Y_a^* \). More precisely, we consider the operator \( \tilde{G}_a \) defined by

\[ (\tilde{G}_a f)(x) = \int_{Y_a^*} \tilde{G}_a(x, y) f(y) \, dy, \]
where $\tilde{G}_a(x,y)$ is the Green function associated with Laplace plus identity operator in the domain $Y^*_a$, with $Y$-periodicity conditions and Dirichlet boundary conditions on $\partial B(0,a)$.

To deduce the relations between the eigenvalues of the operators $\tilde{G}_a$ and $G_a$, we use Lemma 3.9 and the min-max principle. Thus, each eigenvalue $\tilde{\mu}(a)$ of $\tilde{G}_a$ is bounded by an eigenvalue of $G_a$ as follows:

$$\tilde{\mu}^*(a + Ca^2) \leq \tilde{\mu}(a) \leq \tilde{\mu}^*(a - Ca^2).$$

By using this inequality in the estimate (3.17), we get that

$$\mu_1 - K(a + Ca^2)^{-1}\mu_1^2\phi_1^2(0) - O(a^{3/2}/2) \leq \tilde{\mu}(a) \leq \mu_1 - K(a - Ca^2)^{-1}\mu_1^2\phi_1^2(0) + O(a^{3/2}).$$

This last expression can be written as follows:

$$\mu_1 - K(a)^{-1}\mu_1^2\phi_1^2(0) - O(a^{3/2}/2) \leq \tilde{\mu}(a) \leq \mu_1 - K(a)^{-1}\mu_1^2\phi_1^2(0) + O(a^{3/2}). \quad (3.18)$$

Passing to the limit in the previous expression, as $a$ goes to zero, we have that $\lim_{a \to 0} \tilde{\mu}(a) = \mu_1$. Since $\mu_1$ is a simple eigenvalue of the operator $G$, we deduce that, for any $a$ small enough, $\tilde{\mu}(a)$ is the first eigenvalue of $\tilde{G}_a$, that is, $\tilde{\mu}(a) = \frac{1}{1 + \lambda_1(a)}$.

Due to the inequality (3.18) and the fact that $\mu_1 = 1$, we deduce that

$$\frac{1}{1 - K(a)^{-1}\phi_1^2(0) + O(a^{3/2})} \leq 1 + \lambda_1(a) \leq \frac{1}{1 - K(a)^{-1}\phi_1^2(0) - O(a^{3/2})},$$

that is,

$$\lambda_1(a) - K(a)^{-1}\phi_1^2(0) = O(a^{3/2}),$$

and thus the proof of Theorem 3.3 is finished. \hfill \Box

## 4 Bloch waves decomposition

In this section, we introduce the Bloch coefficients for a fixed periodic network of holes which depend on a vector parameter $\eta$ and we study some regularity properties with respect to this parameter. Latter, we present the Bloch waves for a periodic holes network of periodicity dependent on $\varepsilon$. Finally, we prove some results, as the micro-structure size $\varepsilon$ goes to zero, which are necessary for the proof of main result.

### 4.1 Bloch waves in a periodically perforated domain

The Bloch waves method consists in introducing a family of spectral problems parameterized by $\eta \in \mathbb{R}^N$: for each $\eta \in \mathbb{R}^N$, find $\lambda(a;\eta) \in \mathbb{R}$ and $\psi(a;\eta;\eta) \neq 0$ such that

$$\left\{ \begin{array}{ll}
-\Delta \psi(a;\cdot;\eta) = \lambda(a;\eta)\psi(a;\cdot;\eta) & \text{in } \mathbb{R}^N \setminus \overline{S^a}, \\
\psi(a;\cdot;\eta) = 0 & \text{on } \partial S^a, \\
\psi(a;\cdot;\eta) \text{ is } (\eta;Y)-\text{periodic,} \end{array} \right. \quad (4.1)$$

where the condition $\psi(a;\cdot;\eta)$ is $(\eta;Y)$-periodic means

$$\psi(a; y + 2\pi m; \eta) = e^{2\pi i m \cdot \eta} \psi(a; y; \eta) \quad \forall m \in \mathbb{Z}^N, \ y \in \mathbb{R}^N, \ \eta \in \mathbb{R}^N.$$
Since the \((\eta; Y)\)-periodic condition is invariant under translations by elements of \(\mathbb{Z}^N\) in the \(\eta\)-variable, \(\eta\) may be restricted to the set \(Y' = [-\frac{1}{2}, \frac{1}{2}]^N\) which is referred to as the dual cell or Brillouin zone. In the sequel, without loss of generality, we shall take \(\eta \in Y'\).

The solutions of (4.1) are the product of \(Y\)-periodic function with plane wave, i.e.,

\[
\begin{align*}
\psi(a; y; \eta) &= e^{i\eta y} \phi(a; y; \eta), \\
\phi(a; \cdot; \eta) &= \text{is } Y\text{-periodic.}
\end{align*}
\]

Therefore, the problem (4.1) shall be transformed in the following one: for each \(\eta \in Y'\), find \(\lambda(a; \eta) \in \mathbb{R}\) and \(\phi(a; y; \eta) \neq 0\), such that

\[
\begin{align*}
A(\eta)\phi(a; \cdot; \eta) &= \lambda(a; \eta)\phi(a; \cdot; \eta) \quad \text{in } \mathbb{R}^N \setminus \overline{S^a}, \\
\phi(a; \cdot; \eta) &= 0 \quad \text{on } \partial S^a, \\
\phi(a; \cdot; \eta) &= \text{is } Y\text{-periodic},
\end{align*}
\]

where the operator \(A(\eta)\) is the so-called shifted operator, and is given by

\[
A(\eta) = -(\Delta + 2i\eta \cdot \nabla - |\eta|^2).
\]

It is well known that for each fixed \(\eta \in Y'\), the above spectral problem (4.3) admits a countable sequence of positive eigenvalues, each of them having finite multiplicity. As usual, we arrange them in increasing order repeating each eigenvalue according to its multiplicity:

\[
0 < \lambda_1(a; \eta) \leq \lambda_2(a; \eta) \leq \ldots \leq \lambda_m(a; \eta) \leq \ldots \rightarrow \infty.
\]

Besides, the corresponding eigenvectors denoted by \(\{\phi_m(a; \cdot; \eta)\}_{m \geq 1}\), can be chosen to form an orthonormal basis of the space \(L^2_{\#}(Y^*_a)\). It is worthwhile to remark that these eigenvectors belong in fact to the space \(H^1_{0,\#}(Y^*_a)\).

Due to the above parameterized families of eigenvectors and eigenvalues (referred to as Bloch waves), one can describe the spectral resolution of the Laplace operator as an unbounded self-adjoint operator in the space \(L^2(\mathbb{R}^N \setminus \overline{S^a})\). More precisely, we have the following classical result:

**Theorem 4.1.** Let \(g \in L^2(\mathbb{R}^N \setminus \overline{S^a})\) be an arbitrary function. For each \(m \in \mathbb{N}^*\), we define the \(m\)th Bloch coefficient of \(g\) as follows:

\[
(B_m g)(\eta) = \int_{\mathbb{R}^N \setminus \overline{S^a}} g(y) e^{-i\eta y} \overline{\phi_m(a; y; \eta)} \, dy \quad \forall \eta \in Y'.
\]

Then the following inverse formula holds:

\[
g(y) = \int_{Y'} \sum_{m=1}^{\infty} (B_m g)(\eta) e^{i\eta y} \phi_m(a; y; \eta) d\eta.
\]

Further, we have Plancherel identity: for each \(g, h \in L^2(\mathbb{R}^N \setminus \overline{S^a})\),

\[
\int_{\mathbb{R}^N \setminus \overline{S^a}} g(y) \overline{h(y)} \, dy = \int_{Y'} \sum_{m=1}^{\infty} (B_m g)(\eta) \overline{(B_m h)(\eta)} \, d\eta,
\]

and, in particular, Parseval identity:

\[
\int_{\mathbb{R}^N \setminus \overline{S^a}} |g(y)|^2 \, dy = \int_{Y'} \sum_{m=1}^{\infty} |(B_m g)(\eta)|^2 \, d\eta.
\]

The above result is classical in Bloch waves theory and a proof can be found, for instance, in Bensoussan, Lions and Papanicolaou [3, pag. 614].
4.2 Regularity of eigenvalues and eigenvectors

We are interested in the Bloch waves regularity with respect to the parameter \( \eta \) in a neighborhood of the origin, independent of the hole radius \( a \). For this purpose, let us begin by giving the following lower bound:

**Lemma 4.2.** For any \( a > 0 \) small enough, we have that

\[
\lambda_m(a; \eta) \geq \frac{1}{2} \lambda_2^{(N)} > 0 \quad \text{for all} \quad \eta \in Y', m \geq 2,
\]

where \( \lambda_2^{(N)} \) is the second eigenvalue of the Laplace operator in \( Y \) with Neumann boundary conditions.

**Proof.** For all \( m \geq 2 \), we have that \( \lambda_m(a; \eta) \geq \lambda_2(a; \eta) \ \forall \eta \in Y' \). Using the min-max principle, it follows that

\[
\lambda_2(a; \eta) = \min_{\substack{w \subset H^2_0(Y^*_a) \ \dim W = 2 \ \
\text{max } \phi \in W}} \frac{\int_{Y^*_a} |\nabla (e^{i \eta \cdot \phi})|^2 \, dy}{\int_{Y^*_a} |\phi|^2 \, dy} \geq \min_{\substack{w \subset H^2_0(Y^*_a) \ \dim W = 2 \ \
\text{max } \psi \in W}} \frac{\int_{Y^*_a} |\nabla \psi|^2 \, dy}{\int_{Y^*_a} |\psi|^2 \, dy} = \lambda_2^{(N)}(a) \quad \text{for all} \quad \eta \in Y',
\]

where \( \lambda_2^{(N)}(a) \) is the second eigenvalue of the following spectral problem:

\[
\begin{cases}
-\Delta \Psi(a; \cdot) = \lambda^{(N)}(a) \Psi(a; \cdot) & \text{in } Y_a^*, \\
\Psi(a; \cdot) = 0 & \text{on } \partial \mathcal{B}(0; a), \\
\frac{\partial \Psi}{\partial n}(a; \cdot) = 0 & \text{on } \partial Y,
\end{cases}
\]

and the space \( H^1_0(Y^*_a) \) is defined by

\[
H^1_0(Y^*_a) = \{ \Psi \in H^1(Y^*_a) \mid \Psi = 0 \text{ on } \partial \mathcal{B}(0; a) \}.
\]

Following the same approach as in the proof of Theorem 3.1 (see also Rauch and Taylor [20]), we obtain \( \lambda_2^{(N)}(a) \longrightarrow \lambda_2^{(N)} \), as \( a \to 0 \), then for any \( a \) small enough, we conclude

\[
\lambda_2^{(N)}(a) \geq \frac{1}{2} \lambda_2^{(N)} > 0.
\]

The positivity of \( \lambda_2^{(N)} \) is a consequence of the fact that the first eigenvalue \( \lambda_1^{(N)} \) is geometrically simple. 

Now, let notice that in spite of the polynomially dependence on \( \eta \) of the operator \( A(\eta) \), it is well known that the eigenvalues could not be, in general, smooth functions of \( \eta \in Y' \), because of the possible change in the multiplicity. For this reason, we need to study in our case the regularity with respect to \( \eta \in Y' \). Firstly, we prove the following regularity result:
Proposition 4.3. For all $m \geq 1$ and $a > 0$ small enough, the eigenvalues $\lambda_m(a; \eta)$ are Lipschitz functions with respect to $\eta \in Y^1$, where the Lipschitz constant is independent of hole radius $a$.

Proof. We recall that the quadratic form associated with the operator $A(\eta)$ is

$$b^a(\eta; \phi; \phi) = \int_{Y^*_a} \left[ |\nabla \phi|^2 - 2i(\eta \cdot \nabla \phi) \bar{\phi} + |\eta|^2 |\phi|^2 \right] \, dy,$$

for all $\phi \in H^1_{0,H}(Y^*_a)$. We observe that we can decompose the quadratic form $b^a(\eta; \phi; \phi)$ as follows:

$$b^a(\eta; \phi; \phi) = b^a(\eta'; \phi; \phi) + R(\eta, \eta'; \phi, \phi),$$

where

$$R(\eta, \eta'; \phi, \phi) = \int_{Y^*_a} \left[ 2i ((\eta - \eta') \cdot \nabla \phi) \bar{\phi} + (|\eta|^2 - |\eta'|^2) |\phi|^2 \right] \, dy.$$

In order to estimate the term $R(\eta, \eta'; \phi, \phi)$, first we use the Cauchy-Schwarz inequality and we get

$$|R(\eta, \eta'; \phi, \phi)| \leq 2|\eta - \eta'| \|\nabla \phi\|_{L^2(Y^*_a)} \|\phi\|_{L^2(Y^*_a)} + \left( |\eta|^2 - |\eta'|^2 \right) \|\phi\|_{L^2(Y^*_a)}^2.$$ 

Then, there exists a positive constant $C$, independent of $a$, such that

$$|R(\eta, \eta'; \phi, \phi)| \leq C|\eta - \eta'|\|\phi\|_{H^1_{0,H}(Y^*_a)}^2.$$

By the min-max characterization of the eigenvalues we have

$$\lambda_m(a; \eta) = \min_{\substack{W \subset H^1_{0,H}(Y^*_a) \\
\dim W = m}} \max_{\phi \in W \setminus \{0\}} \frac{b^a(\eta; \phi; \phi)}{\|\phi\|_{L^2(Y^*_a)}^2},$$

therefore, due to the previous estimate of the rest $R(\eta, \eta'; \phi, \phi)$, it follows that

$$\lambda_m(a; \eta) \leq \lambda_m(a; \eta') + C|\eta - \eta'| \mu_m(a),$$

(4.6)

where $\mu_m(a)$ is the $m^{th}$ eigenvalue of the following spectral problem:

$$\begin{cases} 
-\Delta \zeta(a; \cdot) + \zeta(a; \cdot) = \mu(a)\zeta(a; \cdot) & \text{in } \mathbb{R}^N \setminus \overline{S^a}, \\
\zeta(a; \cdot) = 0 & \text{on } \partial \mathcal{B}(0, a), \\
\zeta(a; \cdot) \text{ is } Y\text{-periodic}. 
\end{cases}$$

On the other hand, we know that $\mu_m(a)$ converges to the $m^{th}$ eigenvalue $\mu_m$ of the spectral problem associated with Laplace plus identity operator with periodicity conditions (see the proof of Theorem 3.1). More precisely, we have $\mu_m(a) \to \mu_m$, as $a \to 0$. Then, for any small enough $a$, we deduce that

$$\mu_m(a) < 1 + \mu_m = C_1,$$

where $C_1$ is a positive constant independent of $a$. Thus, interchanging $\eta$ and $\eta'$ in (4.6), we obtain that

$$|\lambda_m(a; \eta) - \lambda_m(a; \eta')| \leq C_2 |\eta - \eta'|,$$

where the constant $C_2$ is positive and independent of hole radius. \qed
The Lipschitz property of the first eigenvalue proved above is not enough for the homogenization process. For this reason, we need some analyticity properties. Using perturbation theory, one can prove the following classical analyticity result (for details, see Rellich [21, pag. 57]).

**Proposition 4.4.** For any \( a > 0 \) there exists \( \delta_a > 0 \) such that \( \lambda_1(a; \eta) \) is an analytic function with respect to \( \eta \) in the origin neighborhood \( B_{\delta_a} = \{ \eta : |\eta| < \delta_a \} \). Moreover, the eigenvector \( \phi_1(a; \cdot; \eta) \in H^1_0,*(Y^*) \) is an analytic function with respect to \( \eta \) in \( B_{\delta_a} \).

The inconvenient of the above Proposition is that the neighborhood \( B_{\delta_a} \) is dependent on the hole radius \( a \). In the sequel, we shall prove that there exists a neighborhood independent of the radius \( a \), where the first eigenvalue is analytic with respect to \( \eta \).

**Lemma 4.5.** There exists a neighborhood of \( \eta = 0 \), denoted by \( B_{\delta_1} \), independent of \( a \), such that for all \( a \) small enough, \( \lambda_1(a; \eta) \) is an analytic function with respect to \( \eta \) in \( B_{\delta_1} \).

**Proof.** It is enough to prove that there exists a neighborhood of \( \eta = 0 \), independent of \( a \), where \( \lambda_1(a; \eta) \) is simple for all \( a \) small enough. We begin observing that

\[
|\lambda_1(a; \eta) - \lambda_2(a; \eta)| \geq |\lambda_1(a; \eta) - \lambda_2(a; 0)| - |\lambda_2(a; \eta) - \lambda_2(a; 0)| \\
\geq |\lambda_2(a; 0)| - |\lambda_1(a; \eta) - \lambda_1(a; 0)| - |\lambda_1(a; 0)| - |\lambda_2(a; \eta) - \lambda_2(a; 0)|.
\]

We know that \( \lambda_1(a; 0) \) goes to zero as \( a \to 0 \), then for all \( a \) small enough, we deduce

\[
\lambda_1(a; 0) < \frac{1}{4}\lambda_2^{(N)}.
\]

Since \( \lambda_1(a; \cdot) \) and \( \lambda_2(a; \cdot) \) are Lipschitz functions, with the Lipschitz constant independent of \( a \), and from Lemma 4.2, it follows that

\[
|\lambda_1(a; \eta) - \lambda_2(a; \eta)| \geq \frac{1}{4}\lambda_2^{(N)} - C|\eta|,
\]

for all \( a \) small enough and for any \( \eta \in Y' \). Then, if \( \delta < \lambda_2^{(N)}/(4C) \), we deduce that for all \( |\eta| < \delta \) and \( a \) small enough,

\[
|\lambda_1(a; \eta) - \lambda_2(a; \eta)| > 0.
\]

\( \square \)

### 4.3 Bloch waves at the \( \varepsilon \)-scale

In this subsection, let us introduce the Bloch waves at \( \varepsilon \)-scale and the corresponding Bloch waves decomposition result.

To simplify matters, the unit reference cell \( Y_{\varepsilon}^{*} \), will be denoted by \( Y^{*} \), that is,

\[
Y_{\varepsilon}^{*} = Y \setminus \mathcal{B}(0, \frac{r_{\varepsilon}}{\varepsilon}).
\]

Let us denote by \( \{\lambda_m^{\varepsilon}(r_{\varepsilon}; \xi)\}_{m \geq 1} \) and \( \{\phi_m^{\varepsilon}(r_{\varepsilon}; x; \xi)\}_{m \geq 1} \) the Bloch eigenvalues and eigenvectors at \( \varepsilon \)-scale. This eigenpair gives the diagonalization of the Laplace operator with homogeneous
Dirichlet conditions on the holes boundary $\partial T^\varepsilon$ in the space $L^2 (\mathbb{R}^N \setminus \overline{T})$. By homothecy, for all $m \in \mathbb{N}^*$, we have the following relations:

\[
\begin{align*}
\lambda_m^\varepsilon (r(\varepsilon); \xi) &= \varepsilon^{-2} \lambda_m \left( \frac{r(\varepsilon)}{\varepsilon}; \eta \right), \\
\phi_m^\varepsilon (r(\varepsilon); x; \xi) &= \phi_m \left( \frac{r(\varepsilon)}{\varepsilon}; y; \eta \right),
\end{align*}
\]

where $\{\lambda_m\}_{m \geq 1}$ and $\{\phi_m\}_{m \geq 1}$ are the Bloch eigenvalues and eigenvectors that have already introduced in Subsection 4.1, and the variables $(x; \xi)$ and $(y; \eta)$ are linked by

\[y = \frac{x}{\varepsilon} \quad \text{and} \quad \eta = \varepsilon \xi.\]

We note that $\phi_m^\varepsilon (r(\varepsilon); \cdot; \xi)$ is $\varepsilon Y$-periodic and that $\psi_m^\varepsilon (r(\varepsilon); \cdot; \xi)$ is $(\varepsilon^{-1} \eta; \varepsilon Y)$-periodic.

The following Bloch decomposition result at $\varepsilon$-scale is classical (see Bensoussan, Lions and Papanicolaou [3]) and this theorem is analogous to Theorem 4.1 enunciated in Subsection 4.1.

**Theorem 4.6.** Let $g \in L^2(\mathbb{R}^N \setminus \overline{T})$ be an arbitrary function. For each $m \in \mathbb{N}^*$, the $m$th Bloch coefficient of $g$ at $\varepsilon$-scale is defined by

\[
(B_m^\varepsilon g) (\xi) = \int_{\mathbb{R}^N \setminus \overline{T}} g(x) e^{-i\xi x} \overline{\phi_m^\varepsilon (r(\varepsilon); x; \xi)} \, dx,
\]

for all $\xi \in \varepsilon^{-1} Y'$. Then, the following inverse formula holds:

\[
g(x) = \int_{\varepsilon^{-1} Y'} \sum_{m=1}^{\infty} (B_m^\varepsilon g) (\xi) e^{i\xi x} \phi_m^\varepsilon (r(\varepsilon); x; \xi) \, d\xi.
\]

Further, we have Plancherel identity: $\forall g, h \in L^2(\mathbb{R}^N \setminus \overline{T})$,

\[
\int_{\mathbb{R}^N \setminus \overline{T}} g(x) h(x) \, dx = \int_{\varepsilon^{-1} Y'} \sum_{m=1}^{\infty} (B_m^\varepsilon g) (\xi) (B_m^\varepsilon h)(\xi) \, d\xi,
\]

and, in particular, Parseval identity:

\[
\int_{\mathbb{R}^N \setminus \overline{T}} |g(x)|^2 \, dx = \int_{\varepsilon^{-1} Y'} \sum_{m=1}^{\infty} |(B_m^\varepsilon g) (\xi)|^2 \, d\xi.
\]

### 4.4 Some properties of Bloch waves as $\varepsilon \to 0$

In order to use the spectral method for the homogenization process, we need some properties of the Bloch waves at $\varepsilon$-scale, as $\varepsilon$ goes to zero. This Subsection is devoted to the proof of all these properties.

The following property asserts that, in the limit process, all superior Bloch modes (with $m \geq 2$) can be neglected. Thus, only the first Bloch mode contains the relevant properties for the homogenization process.

**Proposition 4.7.** Let $f \in L^2(\mathbb{R}^N)$ and $v^\varepsilon$ the solution of the following problem

\[
\begin{align*}
-\Delta v^\varepsilon &= f \quad \text{in} \quad \mathbb{R}^N \setminus \overline{T}, \\
v^\varepsilon &= 0 \quad \text{on} \quad \partial T^\varepsilon.
\end{align*}
\]
We define

\[ w^\varepsilon(x) = \int_{\varepsilon^{-1}Y'} \sum_{m=2}^{\infty} (B_m^\varepsilon v^\varepsilon)(\xi) e^{ix\cdot \xi} \varphi_m^\varepsilon(r(\varepsilon); x; \xi) \, d\xi, \]

then, there exists \( C > 0 \), independent of \( \varepsilon \), such that

\[ \|w^\varepsilon\|_{L^2(\mathbb{R}^N \setminus \mathbb{T}^N)} \leq C \varepsilon \|\nabla v^\varepsilon\|_{L^2(\mathbb{R}^N \setminus \mathbb{T}^N)}. \tag{4.8} \]

**Proof.** Since \( v^\varepsilon \) is the solution of (4.7), the Plancherel identity holds and \( (B_m^\varepsilon(-\Delta v^\varepsilon))(\xi) = \lambda_m^\varepsilon(r(\varepsilon); \xi) \lambda_m^\varepsilon(B_m^\varepsilon v^\varepsilon)(\xi) \), we deduce

\[
\int_{\mathbb{R}^N \setminus \mathbb{T}^N} |\nabla v^\varepsilon|^2 \, dx = \int_{\mathbb{R}^N \setminus \mathbb{T}^N} (-\Delta v^\varepsilon) \, v^\varepsilon \, dx = \int_{\varepsilon^{-1}Y'} \sum_{m=2}^{\infty} \lambda_m^\varepsilon(r(\varepsilon); \xi) \lambda_m^\varepsilon(B_m^\varepsilon v^\varepsilon)(\xi) \, d\xi.
\]

By the definition of \( w^\varepsilon \) and the Parseval identity, we get

\[
\|w^\varepsilon\|_{L^2(\mathbb{R}^N \setminus \mathbb{T}^N)}^2 = \int_{\varepsilon^{-1}Y'} \sum_{m=2}^{\infty} |(B_m^\varepsilon v^\varepsilon)(\xi)|^2 \, d\xi \leq \sup_{m \geq 2} \left( \frac{1}{\lambda_m^\varepsilon(r(\varepsilon); \xi)} \right) \int_{\varepsilon^{-1}Y'} \sum_{m=2}^{\infty} \lambda_m^\varepsilon(r(\varepsilon); \xi) |(B_m^\varepsilon v^\varepsilon)(\xi)|^2 \, d\xi.
\]

We have that the right hand side integral is bounded by \( \|\nabla v^\varepsilon\|_{L^2(\mathbb{R}^N \setminus \mathbb{T}^N)} \). On the other hand, \( \lambda_m^\varepsilon(r(\varepsilon); \xi) = \varepsilon^{-2} \lambda_m \left( \frac{r(\varepsilon)}{\varepsilon}; \eta \right) \), then using Lemma 4.2, we obtain that

\[
\frac{1}{\lambda_m^\varepsilon(r(\varepsilon); \xi)} \leq \frac{2 \varepsilon^2}{\lambda_{2^N}^\varepsilon} \quad \forall \xi \in \varepsilon^{-1}Y', \, \forall m \geq 2.
\]

Thus, we get the estimate (4.8), with the constant \( C = \left( \frac{2}{\lambda_{2^N}^\varepsilon} \right)^{1/2} \). \( \square \)

To simplify matters, we can choose an adequate eigenvector \( \varphi_1 \left( \frac{r(\varepsilon)}{\varepsilon}; \cdot; 0 \right) \).

**Proposition 4.8.** There exists a choice of the first eigenvector \( \varphi_1 \left( \frac{r(\varepsilon)}{\varepsilon}; \cdot; 0 \right) \) such that it is real and it satisfies

\[
\tilde{\varphi}_1 \left( \frac{r(\varepsilon)}{\varepsilon}; \cdot; 0 \right) \rightarrow (2\pi)^{-N/2} \text{ strongly in } H^1(Y), \quad \text{as } \varepsilon \to 0, \tag{4.9}
\]

where \( \tilde{\varphi}_1 \) is the extension by zero of \( \varphi_1 \) inside of the ball centered in the origin and radius \( \frac{r(\varepsilon)}{\varepsilon} \).

**Proof.** The choice of \( \varphi_1 \left( \frac{r(\varepsilon)}{\varepsilon}; \cdot; 0 \right) \) as a real number is evident due to the fact that \( \lambda_1 \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) \) is a simple eigenvalue. Moreover, since the normalized first Bloch eigenvector satisfies

\[
\left\| \nabla \varphi_1 \left( \frac{r(\varepsilon)}{\varepsilon}; \cdot; 0 \right) \right\|_{L^2(Y')^N} \to 0, \tag{4.10}
\]

it follows that there exists a constant \( L \) such that \( \tilde{\varphi}_1 \left( \frac{r(\varepsilon)}{\varepsilon}; \cdot; 0 \right) \to L \text{ strongly in } H^1(Y) \). It is clear that \( L = (2\pi)^{-N/2} \). \( \square \)
Now, we study the convergence of the first Bloch mode.

**Proposition 4.9.** Let \( g \in L^2(\mathbb{R}^N) \) and \( g^\varepsilon \) be a sequence in \( L^2(\mathbb{R}^N \setminus T^\varepsilon) \). We denote by \( \tilde{g}^\varepsilon \) the extension by zero inside of the holes \( T^\varepsilon \) of \( g^\varepsilon \), \( B_1^\varepsilon g^\varepsilon \) the first Bloch coefficient of \( g^\varepsilon \) and \( \hat{g} \) the usual Fourier transform of \( g \). If \( \tilde{g}^\varepsilon \to g \) weakly in \( L^2(\mathbb{R}^N) \), then

\[
B_1^\varepsilon \tilde{g}^\varepsilon \to \hat{g} \quad \text{weakly in} \quad L^2_{\text{loc}}(\mathbb{R}^N),
\]

provided there is a fixed compact set \( K \) such that \( \text{supp} \, g^\varepsilon \subseteq K \), for all \( \varepsilon \).

**Proof.** We have that

\[
(B_1^\varepsilon g^\varepsilon)(\xi) = \int_{\mathbb{R}^N \setminus T^\varepsilon} g^\varepsilon(x) e^{-ix \cdot \xi} \overline{\phi_1^\varepsilon(r(\varepsilon); x; \xi)} \, dx
\]

\[
= \int_{\mathbb{R}^N} \tilde{g}^\varepsilon(x) e^{-ix \cdot \xi} \overline{\phi_1^\varepsilon(r(\varepsilon); x; \xi)} \, dx = (B_1^\varepsilon \tilde{g}^\varepsilon)(\xi),
\]

where \( \tilde{\phi_1}(r(\varepsilon); x; \xi) \) is the extension by zero inside of the holes of \( \phi_1^\varepsilon(r(\varepsilon); x; \xi) \). Since \( \text{supp} \, g^\varepsilon \subseteq K \), we write

\[
(B_1^\varepsilon \tilde{g}^\varepsilon)(\xi) = \int_K \tilde{g}^\varepsilon(x) e^{-ix \cdot \xi} \left[ \tilde{\phi_1}^\varepsilon(r(\varepsilon); x; 0) - (2\pi)^{-N/2} \right] \, dx
\]

\[
+ \int_{\mathbb{R}^N} \tilde{g}^\varepsilon(x) e^{-ix \cdot \xi} (2\pi)^{-N/2} \, dx
\]

\[
+ \int_K \tilde{g}^\varepsilon(x) e^{-ix \cdot \xi} \left[ \tilde{\phi_1}^\varepsilon(r(\varepsilon); x; \xi) - \tilde{\phi_1}^\varepsilon(r(\varepsilon); x; 0) \right] \, dx.
\]

For the first integral, we use Lemma 2.5 from Conca, Orive and Vanninathan [9] and Proposition 4.8 in order to obtain that

\[
\left| \int_K \tilde{g}^\varepsilon(x) e^{-ix \cdot \xi} \left[ \tilde{\phi_1}^\varepsilon(r(\varepsilon); x; 0) - (2\pi)^{-N/2} \right] \, dx \right|
\]

\[
\leq \|\tilde{g}^\varepsilon\|_{L^2(\mathbb{R}^N)} \left\| \tilde{\phi_1}^\varepsilon(\frac{r(\varepsilon)}{\varepsilon}; \cdot; 0) - (2\pi)^{-N/2} \right\|_{L^2(Y)}^2 \to 0, \quad \text{as} \quad \varepsilon \to 0,
\]

which in particular, goes to zero in \( L^\infty(\mathbb{R}^N_\xi) \).

Since \( \tilde{g}^\varepsilon \to g \) weakly in \( L^2(\mathbb{R}^N) \), we get that the second integral converges to the Fourier transform of \( g \) weakly in \( L^2(\mathbb{R}^N) \), i.e.,

\[
\int_{\mathbb{R}^N} \tilde{g}^\varepsilon(x) e^{-ix \cdot \xi} (2\pi)^{-N/2} \, dx \to \hat{g} \quad \text{weakly in} \quad L^2(\mathbb{R}^N_\xi).
\]

On the other hand, using again Lemma 2.5 from Conca, Orive and Vanninathan [9] and the fact that the application \( \eta \mapsto \phi_1(\frac{r(\varepsilon)}{\varepsilon}; \cdot; \eta) \in L^2(\mathbb{Y}_\varepsilon^*) \) is Lipschitz (with the Lipschitz constant independent of \( r(\varepsilon) \)), we obtain

\[
\left| \int_K \tilde{g}^\varepsilon(x) e^{-ix \cdot \xi} \left[ \tilde{\phi_1}^\varepsilon(r(\varepsilon); x; \xi) - \tilde{\phi_1}^\varepsilon(r(\varepsilon); x; 0) \right] \, dx \right|
\]

\[
\leq \|\tilde{g}^\varepsilon\|_{L^2(\mathbb{R}^N)} \left\| \tilde{\phi_1}^\varepsilon(\frac{r(\varepsilon)}{\varepsilon}; \cdot; \xi) - \tilde{\phi_1}^\varepsilon(\frac{r(\varepsilon)}{\varepsilon}; \cdot; 0) \right\|_{L^2(Y)} \leq C\varepsilon |\xi|.
\]
Thus, if $|\xi| \leq M$, we see that
\[ \left| \int_K \tilde{g}^\varepsilon(x) e^{-i\xi x} \left[ \tilde{\phi}_1^\varepsilon(r(\varepsilon); x; \xi) - \tilde{\phi}_1^\varepsilon(r(\varepsilon); x; 0) \right] \, dx \right| \leq C\varepsilon M \rightarrow 0, \]
as $\varepsilon \to 0$, and in particular, converges to zero in $L^\infty_{\text{loc}}(\mathbb{R}_\xi^N)$.

Using all previous convergence we conclude. \hfill \Box

Let us now give some properties of the derivatives of first eigenvalue and eigenvector associated with the problem (4.3), when $a = \frac{r(\varepsilon)}{\varepsilon}$, $\eta = 0$ and $\varepsilon$ goes to zero.

**Proposition 4.10.** For all $k, l = 1, \ldots, N$, we have the following properties:
\[ \frac{\partial \lambda_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) = 0, \tag{4.11} \]
\[ \left\| \frac{\partial \phi_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) \right\|_{L^2(Y^*_\varepsilon)} \rightarrow 0 \quad \text{as} \quad \varepsilon \to 0, \tag{4.12} \]
\[ \frac{1}{2} \frac{\partial^2 \lambda_1}{\partial \eta_k \partial \eta_l} \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) \rightarrow \delta_{kl} \quad \text{as} \quad \varepsilon \to 0. \tag{4.13} \]

**Proof.** Given that the maps $\eta \mapsto \lambda_1 \left( \frac{r(\varepsilon)}{\varepsilon}; \eta \right)$ and $\eta \mapsto \phi_1 \left( \frac{r(\varepsilon)}{\varepsilon}; y; \eta \right)$ are smooth for any small enough $\frac{r(\varepsilon)}{\varepsilon} > 0$ (see Proposition 4.4), it is straightforward to compute their derivatives at $\eta = 0$. We differentiate the first eigenvalue equation with respect to $\eta_k$ and we get
\[ \left[ \frac{\partial A(\eta)}{\partial \eta_k} - \frac{\partial \lambda_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon}; \eta \right) \right] \phi_1 \left( \frac{r(\varepsilon)}{\varepsilon}; y; \eta \right) + \left[ A(\eta) - \lambda_1 \left( \frac{r(\varepsilon)}{\varepsilon}; \eta \right) \right] \frac{\partial \phi_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon}; y; \eta \right) = 0. \tag{4.14} \]
Multiplying this equation by $\overline{\phi}_1 \left( \frac{r(\varepsilon)}{\varepsilon}; y; \eta \right)$, integrating by parts and evaluating at $\eta = 0$, it follows that
\[ \frac{\partial \lambda_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) = i \int_{Y^*_\varepsilon} \frac{\partial}{\partial y_k} \left( \frac{r(\varepsilon)}{\varepsilon}; y; 0 \right) \right)^2 \, dy = 0. \]

On the other hand, we observe that $\frac{\partial \phi_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right)$ is solution of the following problem:
\[
\begin{cases}
\left[ -\Delta - \lambda_1 \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) I \right] \frac{\partial \phi_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) &= 2i \frac{\partial \phi_1}{\partial y_k} \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) \quad \text{in} \quad Y^*_\varepsilon, \\
\frac{\partial \phi_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) &= 0 \quad \text{on} \quad \partial \mathcal{B} \left( 0, \frac{r(\varepsilon)}{\varepsilon} \right), \\
\frac{\partial \phi_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) &= Y\text{-periodic}.
\end{cases}
\]
Using the variational formulation of the previous problem and the Cauchy-Schwarz inequality, we get
\[
\left\| \nabla \left( \frac{\partial \phi_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) \right) \right\|_{L^2(Y^*_\varepsilon)^N}^2 \quad \text{and} \quad \left\| \frac{\partial \phi_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) \right\|_{L^2(Y^*_\varepsilon)}^2 
\leq 2 \left\| \nabla \phi_1 \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) \right\|_{L^2(Y^*_\varepsilon)^N} \left\| \frac{\partial \phi_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) \right\|_{L^2(Y^*_\varepsilon)}. \tag{4.15} \]
Since the first derivative of the eigenvector $\phi_1$ with respect to $\eta$ is orthogonal to $\phi_1$, we deduce
\[
\left\| \frac{\partial \phi_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon} ; : 0 \right) \right\|_{L^2(Y^*_\varepsilon)} \leq C \left\| \nabla \left( \frac{\partial \phi_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon} ; : 0 \right) \right) \right\|_{L^2(Y^*_\varepsilon)^N},
\]
where $C > 0$, independent of $\varepsilon$, is equal to an uniform bound of the square root of the inverse of the second eigenvalue associated with Laplace operator in the periodic perforated domain. By using the previous inequality in the estimate (4.15), we obtain that
\[
\left( \frac{1}{C^2} - \lambda_1 \left( \frac{r(\varepsilon)}{\varepsilon} ; 0 \right) \right) \left\| \frac{\partial \phi_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon} ; : 0 \right) \right\|_{L^2(Y^*_\varepsilon)} \leq 2 \left\| \nabla \phi_1 \left( \frac{r(\varepsilon)}{\varepsilon} ; : 0 \right) \right\|_{L^2(Y^*_\varepsilon)^N}.
\]
Since $\lambda_1 \left( \frac{r(\varepsilon)}{\varepsilon} ; 0 \right) \to 0$, as $\varepsilon \to 0$, then for all small enough $\frac{r(\varepsilon)}{\varepsilon}$ we have that $\lambda_1 \left( \frac{r(\varepsilon)}{\varepsilon} ; 0 \right) < \frac{1}{2C^2}$, and consequently,
\[
\left\| \frac{\partial \phi_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon} ; : 0 \right) \right\|_{L^2(Y^*_\varepsilon)} \leq 4C^2 \left\| \nabla \phi_1 \left( \frac{r(\varepsilon)}{\varepsilon} ; : 0 \right) \right\|_{L^2(Y^*_\varepsilon)^N}.
\]
Therefore, due to the convergence (4.10), we get (4.12).

We differentiate at $\eta = 0$ the equation (4.14). Hence, multiplying the result by $\phi_1 \left( \frac{r(\varepsilon)}{\varepsilon} ; y ; 0 \right)$ and integrating, we get
\[
\frac{\partial^2 \lambda_1}{\partial \eta_k \partial \eta_l} \left( \frac{r(\varepsilon)}{\varepsilon} ; 0 \right) = 2\delta_{kl} - 2i \int_{Y^*_\varepsilon} \frac{\partial}{\partial y_k} \left( \frac{\partial \phi_1}{\partial \eta_l} \left( \frac{r(\varepsilon)}{\varepsilon} ; y ; 0 \right) \right) \phi_1 \left( \frac{r(\varepsilon)}{\varepsilon} ; y ; 0 \right) dy
\]
\[
- 2i \int_{Y^*_\varepsilon} \frac{\partial}{\partial y_l} \left( \frac{\partial \phi_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon} ; y ; 0 \right) \right) \phi_1 \left( \frac{r(\varepsilon)}{\varepsilon} ; y ; 0 \right) dy.
\]
By the Cauchy-Schwarz inequality, we deduce that
\[
\left| \frac{\partial^2 \lambda_1}{\partial \eta_k \partial \eta_l} \left( \frac{r(\varepsilon)}{\varepsilon} ; 0 \right) - 2\delta_{kl} \right| \leq 2 \left\| \frac{\partial \phi_1}{\partial \eta_l} \left( \frac{r(\varepsilon)}{\varepsilon} ; : 0 \right) \right\|_{L^2(Y^*_\varepsilon)} \left\| \nabla \phi_1 \left( \frac{r(\varepsilon)}{\varepsilon} ; : 0 \right) \right\|_{L^2(Y^*_\varepsilon)^N}
\]
\[
+ 2 \left\| \frac{\partial \phi_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon} ; : 0 \right) \right\|_{L^2(Y^*_\varepsilon)} \left\| \nabla \phi_1 \left( \frac{r(\varepsilon)}{\varepsilon} ; : 0 \right) \right\|_{L^2(Y^*_\varepsilon)^N}.
\]
With this last inequality and the convergence (4.10) and (4.12), we conclude.

5 Proof of the homogenization result

This section is devoted to the proof of Theorem 2.1. Let us start by recalling that $\{\tilde{u}^\varepsilon\}$ is bounded in $H^1_0(\Omega)$, and therefore, we can extract a subsequence, still denoted by $\{\tilde{u}^\varepsilon\}$, such that
\[
\tilde{u}^\varepsilon \rightharpoonup u \text{ weakly in } H^1_0(\Omega), \text{ as } \varepsilon \text{ goes to zero. (5.1)}
\]

Our goal is to identify the limit $u$ as solution of a partial differential equation (the homogenized equation), by using the spectral method of Bloch waves. Since $u^\varepsilon$ is the solution of the problem (2.1), then for any arbitrary $\varphi \in \mathcal{D}(\mathbb{R}^N)$, with supp$\varphi = K \subset \Omega$, we have that $\varphi \tilde{u}^\varepsilon$ satisfies:
\[
\begin{align*}
-\Delta(\varphi \tilde{u}^\varepsilon) &= F^\varepsilon & \text{in } \mathbb{R}^N \setminus \overline{T^\varepsilon}, \\
\varphi \tilde{u}^\varepsilon &= 0 & \text{on } \partial T^\varepsilon,
\end{align*}
\]
where the right hand side is defined by

\[ F^\varepsilon = f\varphi - 2\nabla \varphi \cdot \nabla \tilde{u}^\varepsilon - \Delta \varphi \, \tilde{u}^\varepsilon. \]

We observe that \( \varphi \tilde{u}^\varepsilon \in H^1_0(\mathbb{R}^N \setminus \overline{T^\varepsilon}) \) and \( \text{supp}(\varphi \tilde{u}^\varepsilon) \subseteq K \).

Let apply the Bloch transform to the equation (5.2) in order to reduce it to the following algebraic equations: for all \( m \geq 1 \) and \( \xi \in \varepsilon^{-1}Y' \),

\[
\lambda_m^\varepsilon(r(\varepsilon)\xi) \ B_m^\varepsilon (\varphi \tilde{u}^\varepsilon) (\xi) = B_m^\varepsilon (F^\varepsilon) (\xi).
\]

Since the sequence \( \varphi \tilde{u}^\varepsilon \) is bounded in \( H^1(\mathbb{R}^N \setminus \overline{T^\varepsilon}) \), we apply Proposition 4.7 with \( u^\varepsilon = \varphi \tilde{u}^\varepsilon \) to neglect all the modes corresponding to \( m \geq 2 \). Then, we pass to the limit as \( \varepsilon \) goes to zero, only in the first Bloch equation, i.e.,

\[
\varepsilon^{-2} \lambda_1 \left( \frac{r(\varepsilon)}{\varepsilon}; \varepsilon \xi \right) \ B_1^\varepsilon (\varphi \tilde{u}^\varepsilon) (\xi) = B_1^\varepsilon (F^\varepsilon) (\xi). \tag{5.3}
\]

Lemma 4.5 allows us to write the Taylor expansion of \( \lambda_1 \left( \frac{r(\varepsilon)}{\varepsilon}; \eta \right) \) around \( \eta = 0 \):

\[
\lambda_1 \left( \frac{r(\varepsilon)}{\varepsilon}; \eta \right) = \lambda_1 \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) + \eta_k \frac{\partial \lambda_1}{\partial \eta_k} \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) + \frac{\eta_k \eta_p}{2} \frac{\partial^2 \lambda_1}{\partial \eta_k \partial \eta_p} \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) + R_\varepsilon,
\]

where

\[
R_\varepsilon = \frac{\eta_k \eta_p \eta_p}{3!} \frac{\partial^3 \lambda_1}{\partial \eta_k \partial \eta_p \partial \eta_p} \left( \frac{r(\varepsilon)}{\varepsilon}; \eta^* \right), \quad \eta^* \in B(0, |\eta|).
\]

By the analyticity of the first Bloch eigenvalue (see Lemma 4.5), we obtain

\[
\varepsilon^{-2} R_\varepsilon = O(\varepsilon \xi^3). \tag{5.4}
\]

By the previous computation and Proposition 4.10, the identity (5.3) becomes

\[
\left[ \varepsilon^{-2} \lambda_1 \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) + \frac{\xi_k \xi_l}{2} \frac{\partial^2 \lambda_1}{\partial \eta_k \partial \eta_l} \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) + O(\varepsilon \xi^3) \right] B_1^\varepsilon (\varphi \tilde{u}^\varepsilon) (\xi) = B_1^\varepsilon (F^\varepsilon) (\xi). \tag{5.5}
\]

Now, let us recall the following estimate stated in the proof of Proposition 4.10:

\[
\left| \frac{\partial^2 \lambda_1}{\partial \eta_k \partial \eta_l} \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) - 2\delta_{kl} \right| \leq C \left\| \nabla \phi_1 \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) \right\|_{L^2(Y_\varepsilon^*)}^2,
\]

where \( C > 0 \) is independent of \( \varepsilon \). Moreover, we have

\[
\left\| \nabla \phi_1 \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) \right\|_{L^2(Y_\varepsilon^*)}^2 = \lambda_1 \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right).
\]

With these two previous properties, the equation (5.5) leads us to

\[
\left[ \varepsilon^{-2} \lambda_1 \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) + \delta_{kl} \xi_k \xi_l + O \left( \lambda_1 \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) \right) + O(\varepsilon \xi^3) \right] B_1^\varepsilon (\varphi \tilde{u}^\varepsilon) (\xi) = B_1^\varepsilon (F^\varepsilon) (\xi). \tag{5.6}
\]
In order to pass to the limit in the last identity, we observe that the sequence \( \{ \tilde{u}^\varepsilon \} \) weakly converges to \( u \in H^1_0(\Omega) \), then \( \varphi \tilde{u}^\varepsilon \rightharpoonup \varphi u \) weakly in \( H^1(\mathbb{R}^N) \), as \( \varepsilon \to 0 \). Using Proposition 4.9, we get that

\[
B^\varepsilon_1 (\varphi \tilde{u}^\varepsilon) (\xi) \rightharpoonup \widehat{\varphi u}(\xi) \quad \text{weakly in } L^2_{\text{loc}}(\mathbb{R}^N), \quad \text{as } \varepsilon \to 0.
\]

Moreover, using the same arguments,

\[
B^\varepsilon_1 (F^\varepsilon) (\xi) \rightharpoonup \widehat{F}(\xi) \quad \text{weakly in } L^2_{\text{loc}}(\mathbb{R}^N), \quad \text{as } \varepsilon \to 0.
\]

Here, we have denoted \( F = f\varphi - 2\nabla \varphi \cdot \nabla u - \Delta \varphi \ u \).

All previous limits allow us transforming the identity (5.6) as follows:

\[
\left[ \lim_{\varepsilon \to 0} \left( \varepsilon^{-2} \lambda_1 \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right) \right) + \delta_{kl} \xi_k \xi_l \right] \widehat{\varphi u}(\xi) = \widehat{F}(\xi),
\]

for each \( \xi \in Y' \).

In order to finish the limit process, let us compute the following limit:

\[
\mathcal{J} = \lim_{\varepsilon \to 0} \varepsilon^{-2} \lambda_1 \left( \frac{r(\varepsilon)}{\varepsilon}; 0 \right).
\]

For this purpose, we use the asymptotic behavior of the first eigenvalue given in Theorem 3.3, written independently of the dimension of space. Thus, the limit \( \mathcal{J} \) becomes

\[
\mathcal{J} = \lim_{\varepsilon \to 0} \left[ S_N \varepsilon^{-2} R \left( \frac{r(\varepsilon)}{\varepsilon} \right) \phi^2_{1}(0) + \mathcal{O} \left( \varepsilon^{-2} R \left( \frac{r(\varepsilon)}{\varepsilon} \right)^{3/2} \right) \right].
\]

In order to prove (i) of the Theorem, we assume that

\[
\lim_{\varepsilon \to 0} \varepsilon^{-2} R \left( \frac{r(\varepsilon)}{\varepsilon} \right) = \ell,
\]

where the constant \( \ell \) is strictly positive. Then, \( \mathcal{J} = S_N \ell \phi^2_{1}(0) \), which implies that the homogenized equation in the Fourier space is \( S_N \ell \phi^2_{1}(0) \widehat{\varphi u}(\xi) - \Delta (\varphi u)(\xi) = \widehat{F}(\xi) \), for all \( \xi \). Applying the inverse Fourier transform, we deduce that

\[
\varphi (S_N \ell \phi^2_{1}(0) u - \Delta u) = f \varphi.
\]

Since \( \varphi \) is arbitrary in \( \mathcal{D}(\Omega) \), we get that \( u \) is the unique solution of the problem (2.4).

Now, let us prove (ii). For this purpose, we assume

\[
\lim_{\varepsilon \to 0} \varepsilon^{-2} R \left( \frac{r(\varepsilon)}{\varepsilon} \right) = 0.
\]

With this, the limit \( \mathcal{J} \) is equal to zero, and therefore the homogenized equation in the Fourier space is \( -\Delta (\varphi u)(\xi) = \widehat{F}(\xi) \), for all \( \xi \). Then, \( -\varphi \Delta u = f \varphi \) and since \( \varphi \) is arbitrary in \( \mathcal{D}(\Omega) \), we conclude that \( u \) is the solution of (2.5).

Finally, in order to prove (iii), we assume

\[
\lim_{\varepsilon \to 0} \varepsilon^{-2} R \left( \frac{r(\varepsilon)}{\varepsilon} \right) = +\infty.
\]

In this case, the equation (5.7) is reduced to \( \widehat{\varphi u}(\xi) = 0 \), which implies that \( u = 0 \).

This completes the proof of Theorem 2.1.
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References


