Proof of a conjecture of José L. Rubio de Francia

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Abstract

Given a compact connected abelian group $G$, its dual group $\Gamma$ can be ordered (in a non-canonical way) so that it becomes an ordered group. It is known that, for any such ordering on $\Gamma$ and $p$ in the range $1 < p < \infty$, the characteristic function $\chi_I$ of an interval $I$ in $\Gamma$ is a $p$–multiplier with a uniform bound (independent of $I$) on the corresponding operator $S_I$ on $L^p(G)$. In this note it is shown that, for $1 < p,q < \infty$, there is a constant $C_{p,q}$, independent of $G$ and the particular ordering on $\Gamma$, such that

$$\| (\sum_j |S_I f_j|^q)^{1/q}\|_{L^p(G)} \leq C_{p,q} \| (\sum_j |f_j|^q)^{1/q}\|_{L^p(G)}$$

for all sequences $\{I_j\}$ of intervals in $\Gamma$ and all sequences $\{f_j\}$ in $L^p(G)$. Such a result was conjectured by J.L. Rubio de Francia, who noted its validity when $G = \mathbb{T}^n$. The present proof uses a transference argument, an approach which shows that any constant $C_{p,q}$ for which the inequality holds when $G = \mathbb{T}^n$ will serve for every $G$ and every ordering on $\Gamma$. An added advantage of this approach is that it adapts to give an extension of the result for functions taking values in a UMD space.

1 Introduction

Let $G$ be a compact connected abelian group with dual group $\Gamma$. Then $\Gamma$ can be ordered (in a non-canonical way) so that it becomes an ordered group. Fix any such ordering $\leq$ on $\Gamma$ and let $\Gamma^+ = \{ \gamma \in \Gamma : \gamma \geq 0 \}$. A classical result of Bochner [Bo] asserts that, for $1 < p < \infty$, the characteristic function of $\Gamma^+$ is a $p$–multiplier. It follows immediately that, for each interval $I$ in $\Gamma$, $\chi_I$ is a $p$–multiplier with a uniform bound on its multiplier norm independent of $I$. (The intervals in $\Gamma$ depend, of course, on the particular ordering chosen and may or may not include either of their end-points.)

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Given an interval $I$, let $S_I$ denote the corresponding operator on $L^p(G)$. In [RdeF] J.L.Rubio de Francia observed in the above context that, for $1 < p < \infty$ and $\frac{1}{p} < \frac{2}{q} < \frac{p+1}{p}$, there is a constant $C_{p,q}$ such that

\[(1.1) \quad \| (\sum_j |S_I f_j|^q)^{1/q} \|_{L^p(G)} \leq C_{p,q} \| (\sum_j |f_j|^q)^{1/q} \|_{L^p(G)} \]

for all sequences $\{I_j\}$ of intervals in $\Gamma$ and all sequences $\{f_j\}$ in $L^p(G)$. He noted that, when $G = \mathbb{T}$ or $\mathbb{T}^n$, an inequality of the form (1.1) is in fact valid for all $p, q$ in the range $1 < p, q < \infty$ and conjectured that this would be the case for an arbitrary compact connected abelian $G$. We establish this conjecture using ideas developed in [B,G], [B,G,M] and [B,G,T]. Specifically, we first use a transference argument to deduce the result for $\mathbb{T}^n$ from that for $\mathbb{T}$. Structural considerations then give the result for a general $G$. This approach has the advantage of showing that, if $1 < p, q < \infty$, then any constant $C_{p,q}$ for which the inequality (1.1) holds when $G = \mathbb{T}$ will in fact serve for every $G$ and every ordering in $\Gamma$. In particular, the constant in the inequality for $\mathbb{T}^n$ can be taken to be independent of dimension and of the ordering in $\mathbb{Z}^n$.

We state the result formally as follows.

**Theorem A** Let $1 < p, q < \infty$. Then there is a constant $C_{p,q}$ with the following property. For every compact connected abelian group $G$ with ordered dual $(\Gamma, \leq)$,

\[(1.2) \quad \| (\sum_j |S_I f_j|^q)^{1/q} \|_{L^p(G)} \leq C_{p,q} \| (\sum_j |f_j|^q)^{1/q} \|_{L^p(G)} \]

for all sequences $\{I_j\}$ of intervals in $\Gamma$ and all sequences $\{f_j\}$ in $L^p(G)$. Furthermore, if $\alpha_{p,q}$ is a constant such that

\[(1.3) \quad \| (\sum_j |S_I f_j|^q)^{1/q} \|_{L^p(\mathbb{T})} \leq \alpha_{p,q} \| (\sum_j |f_j|^q)^{1/q} \|_{L^p(\mathbb{T})} \]

holds for all sequences of intervals $\{I_j\}$ in $\mathbb{Z}$ and all sequences $\{f_j\}$ in $L^p(\mathbb{T})$, then we can take $C_{p,q}$ to equal $\alpha_{p,q}$.

**Comment.** In fact, as we shall note at the end of the paper, if $C_{p,q}$ is a constant such that (1.2) holds for a particular (non trivial) group $G$ and a particular ordering $\leq$ of $\Gamma$, then (1.2) will be valid with that same constant when $G = \mathbb{T}$. Consequently, the best constant $C_{p,q}$ in (1.2) for an individual group $G$ (other than the trivial one) will equal the best constant $\alpha_{p,q}$ in (1.3).

A further advantage of our approach is that it adapts to give an extension of the conjecture of Rubio de Francia to functions taking values in a UMD space (see Theorem B at the end of this note). Recall that a Banach space $E$ is in the class UMD if the periodic Hilbert transform is a bounded operator from $L^p_E(\mathbb{T})$ into itself for any (or equivalently all) $p$ in the range $1 < p < \infty$ (see [Bk] and [Bu]).
2 Proof of Theorem A

(i) The case when $G = T$. Here $\Gamma = \mathbb{Z}$ has only two orderings, the natural one and its reverse, and these are isomorphic. Standard arguments give that, in this setting, (1.2) is equivalent to an inequality of the form

$$\| (\sum_j |Pf_j|^q)^{1/q} \|_{L^p(T)} \leq D_{p,q} \| (\sum_j |f_j|^q)^{1/q} \|_{L^p(T)}$$

for all $\{f_j\}$ in $L^p(T)$, where $P$ is the operator corresponding to the $p-$multiplier $\chi_{\{n \in \mathbb{Z}: n \geq 0\}}$.

Now (2.1) follows immediately from the boundedness of the periodic Hilbert transform acting on $L^p_\ell(T)$, a result which appears to have been established first in ([B,B]).

For the remainder of the proof of the theorem, let $\alpha_{p,q}$ be a constant such that

$$\| (\sum_j |S_I f_j|^q)^{1/q} \|_{L^p(T)} \leq \alpha_{p,q} \| (\sum_j |f_j|^q)^{1/q} \|_{L^p(T)}$$

for all sequences of intervals $\{I_j\}$ in $\mathbb{Z}$ and all sequences $\{f_j\}$ in $L^p(T)$.

(ii) A self-improvement of the case $G = T$. Before discussing the case $G = T^n$, we make a further observation about the case $G = T$. Suppose that $(\Omega, \mu)$ is an arbitrary measure space and $I$ is an interval in $\mathbb{Z}$. The operator $S_I$ has a natural extension to the space $L^p_{L^p(\mu)}(T)$, obtained by first defining it as $S_I \otimes Id$ on the algebraic tensor product $L^p(T) \otimes L^p(\mu)$ and noting that, by Fubini’s theorem, this extension retains the operator norm of the initial operator $S_I$ on $L^p(T)$. Use density to extend to all of $L^p_{L^p(\mu)}(T)$ and denote by $S_I$ the resulting operator.

Now let $\{I_j\}$ be a sequence of intervals in $\mathbb{Z}$ and consider $g \in L^p_{L^p(\mu)}(T)$. Then $g$ corresponds to a sequence $\{g_j\}$ where $g_j \in L^p_{L^p(\mu)}(T)$. With $S_I g_j$ defined as above as an element of $L^p_{L^p(\mu)}(T)$, (2.2) and Fubini’s theorem give

$$\| (\sum_j |S_I g_j|^q)^{1/q} \|_{L^p_{L^p(\mu)}(T)} \leq \alpha_{p,q} \| (\sum_j |g_j|^q)^{1/q} \|_{L^p_{L^p(\mu)}(T)}.$$  

(iii) The case when $G = T^n$ and $\Gamma = \mathbb{Z}^n$ has some ordering $\leq$. We obtain the result in this case from the result for $G = T$ and a transference argument. A similar technique was used in the proof of [B,G,M], Theorem 4.1 although we need to modify the transference principle in the present context.

Fix $n$ and some ordering $\leq$ on $\mathbb{Z}^n$. We shall use the notation $\gamma = (k_1, ..., k_n)$ either for a point in $\mathbb{Z}^n$ or for the corresponding character $\gamma(\omega) = \omega_1^{k_1} ... \omega_n^{k_n}$ on $T^n$. We require the following Lemma.

**Lemma 2.4** Let $F$ be a finite subset of $\mathbb{Z}^n$. Then there exists $m \in \mathbb{Z}^n$ such that the function $k \rightarrow m.k$ is strictly increasing on $F$. 

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Proof Assume that $F$ contains more than one point and let $\bar{F} = \{k_1 - k_2 : k_1, k_2 \in F \text{ and } k_1 > k_2\}$. Then $\bar{F}$ is a finite subset of $\{k \in \mathbb{Z}^n : k > 0\}$. It follows that there exists $m \in \mathbb{Z}^n$ such that $m.k > 0$ for all $k \in \bar{F}$ (see [B,G,M] p.10). This gives the required result.

To establish (1.2) for $G = T^n$, we consider first a finite set $f_1, \ldots, f_J$ of trigonometric polynomials of the form

$$(2.5) \quad f_j = \sum_{\gamma \in F_j} \beta_{j,\gamma}\gamma \quad (j = 1, \ldots, J)$$

where $\beta_{j,\gamma} \in \mathbb{C}$ and $F_1, \ldots, F_J$ are finite subsets of $\mathbb{Z}^n$. Let $I_1, \ldots, I_J$ be intervals in $\mathbb{R}$. By Lemma (2.4), there exists $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. By transference result ([C,W],Theorem 2.4) and will be discussed elsewhere [B,G,T]. The proof of more general theorem involving operator-valued kernels that is analogous to the general transference result ([C,W],Theorem 2.4) will be discussed elsewhere [B,G,T].

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We shall apply the following transference result, which is a special case of a somewhat more general theorem involving operator-valued kernels that is analogous to the general transference result ([C,W],Theorem 2.4) and will be discussed elsewhere [B,G,T]. The proof of the simple case required here is included for completeness.

**Operator-valued Transference.** Let $u \rightarrow R_u$ be a strongly continuous representation of a compact abelian group $G$ on a Banach space $X$ and let $c = \sup\{\|R_u\| : u \in G\}$. (Note that $c < \infty$ by strong continuity and the principle of uniform boundedness.) Let $K(u) \in \mathcal{L}(X)$ for $u \in G$ with $K(u)R_v = R_vK(u)$ for all $u, v \in G$ and suppose that $u \rightarrow K(u)$ is Bochner integrable with respect to the operator norm. Let $T_K : X \rightarrow X$ be defined by

$$T_Kx = \int_G K(u)R_{-u}xdu \quad (x \in X).$$

Then, for $1 \leq p < \infty$, $\|T_K\| \leq c^2N_{p,X}(K)$, where $N_{p,X}(K)$ is the norm of the operator-valued convolution defined on $L^p_X(G)$ by $K \ast g(v) = \int_G K(u)g(v-u)du$.

Proof. Fix $x \in X$ and average the inequality

$$\|T_Kx\|_X^p \leq c^p\|R_uT_Kx\|_X^p$$
over \( v \in G \) to obtain
\[
\|T_K x\|_X^p \leq c^p \int_G \|R_v K(u) R_{-u} x\|_X^p \, dv = c^p \int_G \|K(u) R_{-u} x\|_X^p \, dv \\
\leq c^p N_{p,X}(K)^p \int_G \|R_v x\|_X^p \, dv \leq c^p N_{p,X}(K)^p \|x\|_X^p.
\]

Take \( G = T \) and \( X = L_{\ell_q}^p(T^n) \) in this theorem and consider the isometric representation \( u \to R_u \) of \( T \) in \( L_{\ell_q}^p(T^n) \) given by
\[
(R_u \psi)(\omega_1, \ldots, \omega_n) = \psi(u^{m_1} \omega_1, \ldots, u^{m_n} \omega_n),
\]
where \( m = (m_1, \ldots, m_n) \) was used to define \( \tau \) in accordance with Lemma 2.4. Let the kernel \( K \) be defined for \( u \in T \) by
\[
K(u)(\psi) = \{h_j(u) \psi_j\} \text{ where } \psi = \{\psi_j\}.
\]
In the notation of our transference theorem, (2.7) implies that \( N_{p,X}(K) \leq \alpha_{p,q} \). Hence, since \( c = 1 \),
\[
(2.9) \quad \|\int_T K(u) R_{-u} du\| \leq \alpha_{p,q}.
\]
Now
\[
\int_T K(u) R_{-u} \{f_j\} \, du = \{ \int_T h_j(u) \sum_{\gamma \in F_j} \beta_{j,\gamma}(R_{-u} \gamma) du \} = \{ \sum_{\gamma \in F_j} \beta_{j,\gamma} \hat{h}_j(\tau(\gamma)) \gamma \} = \{ S_{I_j} f_j \},
\]
using (2.6), the definition of \( h_j \) and the fact that, if \( \gamma = (k_1, \ldots, k_n) \in \mathbb{Z}^n \), then \( R_{-u} \gamma = u^{-m.k.\gamma} \) for \( u \in T \). Hence (2.9) gives
\[
(2.10) \quad \| \left( \sum_j |S_{I_j} f_j|^q \right)^{1/q} \|_{L^p(T^n)} \leq \alpha_{p,q} \left( \sum_j |f_j|^q \right)^{1/q} \|_{L^p(T^n)}.
\]
Since this is valid for all trigonometric polynomials and all \( J \in \mathbb{N} \), (1.2) is established for \( G = T^n \) with \( C_{p,q} = \alpha_{p,q} \).

(iv) The case when \( G \) is arbitrary and \( \leq \) is any ordering on \( \Gamma \). As in case (iii), consider a finite set \( f_1, \ldots, f_J \) of trigonometric polynomials of the form
\[
(2.11) \quad f_j = \sum_{\gamma \in F_j} \beta_{j,\gamma} \gamma \quad (j = 1, \ldots, J),
\]
where \( \beta_{j,\gamma} \in \mathbb{C} \) and \( F_1, \ldots, F_J \) are finite subsets of \( \Gamma \). Let \( I_1, \ldots, I_J \) be intervals in \( \Gamma \). Denote by \( \Lambda \) the subgroup of \( \Gamma \) generated by \( \bigcup_{j=1}^J F_j \). Then \( \Lambda \) is torsion free (since \( G \) is connected)
and hence is isomorphic to \( \mathbb{Z}^n \) for some \( n \in \mathbb{N} \). Let \( H \) denote the annihilator of \( \Lambda \) in \( G \). Then \( G/H \) has dual group \( \Lambda \) and so \( G/H \) is isomorphic to \( \mathbb{T}^n \). By Weyl’s formula,

\[
(2.12) \quad \int_G \left( \sum_{j=1}^J |f_j|^q \right)^{p/q} du = \int_{G/H} \{ \int_H \left( \sum_{j=1}^J |f_j(u+h)|^q \right)^{p/q} dh \} d(u+H) = \int_{G/H} \left( \sum_{j=1}^J |f_j^#(u+H)|^q \right)^{p/q} d(u+H),
\]

where \( f_j^#(u+H) = f_j(u) \) for \( u+H \in G/H \) and \( dh,d(u+H) \) denote normalized Haar measure on \( H,G/H \) respectively.

The ordering \( \leq \) on \( \Gamma \) restricts to an ordering on \( \Lambda \), and for each \( j \), there is an interval \( I_j^# \) in \( \Lambda \) such that \( I_j^# \cap F_j = I_j \cap F_j \). Furthermore, \( S_{I_j^#} f_j^#(u+H) = (S_{I_j} f_j)^#(u+H) \) for \( u+H \in G/H \). Weyl’s formula also gives

\[
(2.13) \quad \int_G \left( \sum_{j=1}^J |S_{I_j} f_j|^q \right)^{p/q} du = \int_{G/H} \left( \sum_{j=1}^J |S_{I_j} f_j^#(u+H)|^q \right)^{p/q} d(u+H) = \int_{G/H} \left( \sum_{j=1}^J |S_{I_j} f_j^#(u+H)|^q \right)^{p/q} d(u+H).
\]

Identifying \( G/H \) with \( \mathbb{T}^n \), case (iii) gives

\[
\int_{G/H} \left( \sum_{j=1}^J |S_{I_j} f_j^#(u+H)|^q \right)^{p/q} d(u+H) \leq \alpha_{p,q} \int_{G/H} \left( \sum_{j=1}^J |f_j^#(u+H)|^q \right)^{p/q} d(u+H).
\]

Hence

\[
\int_G \left( \sum_{j=1}^J |S_{I_j} f_j|^q \right)^{p/q} du \leq \alpha_{p,q} \int_G \left( \sum_{j=1}^J |f_j|^q \right)^{p/q} du
\]

by (2.12) and (2.13). Since this is valid for all trigonometric polynomials and all \( J \in \mathbb{N} \), we conclude that (1.2) holds with \( C_{p,q} = \alpha_{p,q} \) for all sequences \( \{f_j\} \) in \( L^p(G) \) and all \( \{I_j\} \). This completes the proof of Theorem A.

3 Concluding Remarks

It is worth noting that, if \( C_{p,q} \) is a constant such that (1.2) holds for any individual non-trivial \( G \) and ordering \( \leq \) of \( \Gamma \), then (1.3) will be valid with \( \alpha_{p,q} = C_{p,q} \). To see this, choose a non-zero element \( \gamma_0 \) in \( \Gamma \), let \( \Lambda_0 \) be the subgroup of \( \Gamma \) generated by \( \gamma_0 \) and let \( H_0 \) be the annihilator of \( \Lambda_0 \) in \( G \). Computing as in (2.12) and (2.13) and adopting similar notation, we see that the inequality (1.2) will imply that
\begin{equation}
\int_{G/H_0} \left( \sum_{j=1}^{J} |S_{I^*} f_j^#(u + H_0)|^q \right)^{p/q} d(u + H_0) 
\end{equation}

\leq C_{p,q} \int_{G/H_0} \left( \sum_{j=1}^{J} |f_j^#(u + H_0)|^q \right)^{p/q} d(u + H_0)

whenever $f_1, \ldots, f_J$ are trigonometric polynomials only involving elements of $\Lambda_0$ and $I_1^#, \ldots, I_J^#$ are intervals in $\Lambda_0$. (Note that every interval $I^#$ in $\Lambda_0$ is of the form $I \cap \Lambda_0$ for some interval $I$ in $\Gamma$.) Since $\Lambda_0$ is isomorphic to $\mathbb{Z}$, its dual $G/H_0$ is isomorphic to $\mathbb{T}$. Hence (3.1) implies that (1.3) holds with $\alpha_{p,q}$ replaced by $C_{p,q}$.

It is well known that, if $E$ is a UMD Banach space, the space $\ell^q_E$ is also UMD and in particular the periodic Hilbert transform is bounded on $L^p_{\ell^q_E}(\mathbb{T})$ for $1 < p, q < \infty$. Hence

\begin{equation}
\| \left( \sum_j \| Pf_j \|^q_E \right)^{1/q} \|_{L^p(\mathbb{T})} \leq D_{p,q,E} \| \left( \sum_j \| f_j \|^q_E \right)^{1/q} \|_{L^p(\mathbb{T})}
\end{equation}

for all $\{f_j\}$ in $L^p_{\ell^q_E}(\mathbb{T})$, where $P$ now denotes the operator on $L^p_{\ell^q_E}(\mathbb{T})$ corresponding to the $p$-multiplier $\chi_{\{n\in\mathbb{Z}:n\geq0\}}$. Starting with (3.2), the proof of Theorem A adapts to the setting of vector-valued functions, giving the following result.

**Theorem B** Let $E$ be a UMD Banach space and $1 < p, q < \infty$. Then there is a constant $C_{p,q,E}$ with the following property. For every compact connected abelian group $G$ with ordered dual $(\Gamma, \leq)$,

\begin{equation}
\| \left( \sum_j \| S_{I^*} f_j \|^q_E \right)^{1/q} \|_{L^p(G)} \leq C_{p,q,E} \| \left( \sum_j \| f_j \|^q_E \right)^{1/q} \|_{L^p(G)}
\end{equation}

for all sequences $\{I_j\}$ of intervals in $\Gamma$ and all sequences $\{f_j\}$ in $L^p_E(G)$. Furthermore, if $\alpha_{p,q,E}$ is a constant such that

\begin{equation}
\| \left( \sum_j \| S_{I^*} f_j \|^q_E \right)^{1/q} \|_{L^p(\mathbb{T})} \leq \alpha_{p,q,E} \| \left( \sum_j \| f_j \|^q_E \right)^{1/q} \|_{L^p(\mathbb{T})}
\end{equation}

holds for all sequences $\{I_j\}$ of intervals in $\mathbb{Z}$ and all sequences $\{f_j\}$ in $L^p_E(\mathbb{T})$, then we can take $C_{p,q,E}$ to equal $\alpha_{p,q,E}$.

Moreover, the best constant $\alpha_{p,q,E}$ will equal the best constant $C_{p,q,E}$ in (3.3) for each individual $G$ (other than the trivial group).

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