Math Review

Basic math instruments we will need along the course. References:

- Appendix 2 in Blanchard (2006)

1 Geometric progressions

Definition:

\[ 1 + x + x^2 + x^3 + \ldots + x^n, \]

where \( x \neq 1 \).

Some possible questions:

- how much is the result of the summation of all these terms?
- where does the summation converge to? (Is it finite or infinite?)

Summation:

- **Finite** geometric progression

\[
1 + x + x^2 + x^3 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}
\]

- **Infinite** geometric progression

\[
1 + x + x^2 + x^3 + \ldots = \frac{1}{1 - x},
\]

only if \(|x| < 1\).

2 Approximation

If the numbers of rates we are using are small enough (between 0% and 10%), we can simplify some expressions:

\[
(1 + x)(1 + y) \approx 1 + x + y,
\]

\[
(1 + x)^2 \approx 1 + 2x, \text{ then, } (1 + x)^n \approx 1 + nx,
\]

\[
\frac{1 + x}{1 + y} \approx 1 + x - y.
\]

An example of how we can use this type of approximations, is computing the real interest rate from the nominal interest rate:

\[
(1 + r) = \frac{1 + i}{1 + \pi} \rightarrow r \approx i - \pi.
\]
3 Growth rates

References: Belzunegui et al. Appendix 7.1.

Let us assume the variable $X_t$ varies between $t$ and $t + 1$ as follows:

$$X_{t+1} = X_t + \gamma_t X_t$$

$$= X_t (1 + \gamma_t),$$

where $\gamma_t$ is called the growth rate of $X_t$ between $t$ and $t + 1$.

To compute the growth rate, we just need to solve for $\gamma_t$

$$X_{t+1} = X_t (1 + \gamma_t)$$

$$= X_t (1 + \gamma_t)$$

$$= X_t (1 + \gamma_t)$$

$$\gamma_t = \frac{X_{t+1}}{X_t} - 1 = \frac{X_{t+1} - X_t}{X_t}$$

and we can express it in absolute or percentage terms.

Growth rates can also be constant:

$$X_{t+1} = X_t (1 + \gamma).$$

3.1 Growth rates for more than one period

Let us assume that we know the rate of variation of a series $X_t$ for several periods: \{ $X_t, X_{t+1}, X_{t+2}, \ldots, X_{t+p}$ \}.

We know that

$$X_{t+1} = X_t (1 + \gamma_t),$$

$$X_{t+2} = X_{t+1} (1 + \gamma_{t+1}) = X_t (1 + \gamma_t) (1 + \gamma_{t+1}),$$

$$X_{t+3} = X_{t+2} (1 + \gamma_{t+2}) = X_{t+1} (1 + \gamma_{t+1}) (1 + \gamma_{t+2}) = X_t (1 + \gamma_t) (1 + \gamma_{t+1}) (1 + \gamma_{t+2}),$$

$$\ldots$$

$$X_{t+p} = X_{t+p-1} (1 + \gamma_{t+p-1}) = X_t (1 + \gamma_t) \ldots (1 + \gamma_{t+p-1}).$$

We can also compute the average growth rate

$$X_{t+p} = X_t (1 + \gamma) \ldots (1 + \gamma) = X_t (1 + \gamma)^p.$$

To solve for it if we have $X_{t+p}$ and $X_t$ then

$$X_{t+p} = X_t (1 + \gamma)^p,$$

$$\frac{X_{t+p}}{X_t} = (1 + \gamma)^p,$$

$$\gamma = \left( \frac{X_{t+p}}{X_t} \right)^\frac{1}{p} - 1.$$

For example, to compute the rate of inflation, we solve for the growth rate or rate of variation of prices:

$$\pi_t = \left( \frac{P_t - P_{t-1}}{P_{t-1}} \right) \times 100 = \left[ \frac{P_t}{P_{t-1}} - 1 \right] \times 100$$
4 Present value (or present discounted value)

How to value future income now. We can use the real interest rate:

One cookie in period $t$ → $1 + r_t$ cookies in period $t + 1$

Therefore,

$\frac{1}{1 + r_t}$ cookies in period $t$ → 1 cookie in period $t + 1$

If we save during 2 periods, what we get is

One cookie in $t$ → $1 + r_t$ cookies in $t + 1$ → $(1 + r_t)(1 + r_{t+1})$ cookies in $t + 2$

Therefore,

$\frac{1}{(1 + r_t)(1 + r_{t+1})}$ cookies in $t$ → $\frac{1}{(1 + r_{t+1})}$ cookie in $t + 1$ → 1 in $t + 2$

Example:

20 cookies in $t + 1$ are worth $\frac{20}{1 + r_t}$ cookies in $t$.

Example:

If the interest rate is $r_t = r_{t+1} = \frac{1}{1.1} = 10\%$, then 500 cookies in $t+2$ are worth $\frac{500}{1.1} = 454.55$ cookies in $t$.

Example: One person receives 100 cookies in period $t$, 150 cookies in $t + 1$, and 200 cookies in $t + 2$. Assume that $r_t = 5\%$ and $r_{t+1} = 6\%$. What is the present value of all this income flow in terms of cookies at time $t$?

$$ PV = 100 + \frac{150}{1.05} + \frac{200}{(1.06)(1.05)} = 100 + 142.857 + 179.695 = 422.552 $$

cookies in time $t$.

4.1 Infinite number of periods

In general, if you receive income $w_t$ in period $t$, $w_{t+1}$ in $t + 1$, and so on, then the present value (from the point of view of time $t$) of all this income is:

$$ PV = w_t + \frac{w_{t+1}}{1 + r_t} + \frac{w_{t+2}}{(1 + r_t)(1 + r_{t+1})} + \frac{w_{t+3}}{(1 + r_t)(1 + r_{t+1})(1 + r_{t+2})} + ... $$

(1)

where $r_t$ is the real interest rate between $t$ and $t + 1$.

But if both income and real interest rates are constant, we can simplify the formula. To do that, we need to do as follows:

$$ (1 - x)(1 + x + x^2 + x^3 + ...) = 1 + x + x^2 + x^3 + ... - (x + x^2 + x^3 + x^4 + ...) = 1 $$

(2)

and then, we have one of the key formulas

$$ 1 + x + x^2 + x^3 + ... = \frac{1}{1 - x}. $$

(3)

If the real interest rate $r$ and income $w$ are constant for all the periods, then the present value of all the income flow is

$$ PV = w + \frac{w}{1 + r} + \frac{w}{(1 + r)^2} + ... = w \left( 1 + \frac{1}{1 + r} + \frac{1}{(1 + r)^2} + ... \right) = \frac{w}{1 - \frac{1}{1 + r}}. $$

(4)
that is,

\[ PV = \frac{1 + r}{r} w. \] (5)

4.2 Finite number of periods

If the real interest rate \( r \) is constant, and you receive a constant income flow \( w \) from period 0 to period \( T \), then the present value of that income flow, in terms of period 0, is

\[
P V = w + \frac{w}{1 + r} + \frac{w}{(1 + r)^2} + \ldots + \frac{w}{(1 + r)^T} = \frac{1 + r}{r} w - \frac{w}{(1 + r)^{T+1}} - \frac{w}{(1 + r)^{T+2}} - \ldots, \tag{6}
\]

that is,

\[
P V = \left(1 - \frac{1}{(1 + r)^{T+1}}\right) \frac{1 + r}{r} w. \tag{7}
\]

**Example**: if you earn 1000 cookies every period from \( t = 0 \) to \( t = 10 \), and the real interest rate is \( r = 10\% = 0.1 \), the present value of this income flow is

\[
P V = \left(1 - \frac{1}{(1.1)^{T+1}}\right) \times \frac{11}{0.1} \times 1000 = 7144.567 \text{ cookies at time } t = 0. \tag{8}
\]

4.3 More examples

**Example 1**: To value at time 2006 a bond that pays \( X \) cookies at time 2009

\[
P V = \frac{X}{(1 + r_{2006})(1 + r_{2007})(1 + r_{2008})} \text{ cookies.} \tag{9}
\]

It is worth buying it if the price of the bond is less than the PV.

It is worth selling it if the price of the bond is more than the PV.

We are indifferent between buying or selling it if PV=PRICE.

**Example 2**: Assume that IBM stocks pay (and will always pay) a dividend of 5 cookies a year, and that the real interest rate is (and will be) 4% per year. Then the present value of an IBM stock is

\[
P V = 5 + \frac{5}{1.04} + \frac{5}{(1.04)^2} + \ldots = \frac{5}{1 - \frac{1}{1.04}} = \frac{1.04}{0.04}5 = 130 \text{ cookies.} \tag{10}
\]

Then, it is worth buying the stocks when its price is less than 130.

It is worth selling the stocks when its price is more than 130.

We are indifferent between selling or buying when its price is equal to 130.

**Example 3**: We are thinking of whether to buy an apartment that costs 300,000 cookies, or renting an apartment for 900 cookies/month. The annual interest rate is 6%, an that is why the monthly interest rate is 0.5% = 0.005. Therefore, the present value of renting is

\[
P V = 900 + \frac{900}{1.005} + \frac{900}{(1.005)^2} + \ldots = \frac{1.005}{0.005}900 = 180900 \text{ cookies.} \tag{11}
\]

Question: should we buy or rent?