4.21

a. $PV_0 = \frac{C}{r} = \frac{1,000}{0.1} = 10,000$

b. $PV_0 = \frac{PV_i}{1 + r} = \frac{C}{1 + r} = \frac{500 \times 1}{0.1 \times 1.1} = 4,545.5$

c. $PV_0 = PV_2 \times \frac{1}{1 + r} = \frac{C}{1 + r} = \frac{2,420 \times 1}{0.1 \times 1.1^2} = 20,000$

4.26

Barrett is indifferent between accepting or rejecting the project when the net present value of the project is zero.

$$NPV_0 = -C_0 + \sum_{t=1}^{\infty} \frac{C_t}{(1 + r)^t} = -C_0 + \frac{C_t}{r} = -100,000 + \frac{50,000}{r} = 0$$

Then $r = 50\%$.

4.28

Quarterly interest rate = $15\% / 4 = 3.75\%$

The first payment is 5 years from now, in other words, 20 quarters from now.

$$PV_0 = PV_{19} \times \frac{1}{(1 + r)^{19}} = \frac{C}{r} \times \frac{1}{0.0375 \times 1.0375^{19}} = \frac{1}{1.0375} = 13.25$$

4.29

$$PV_0 = \frac{C}{r} \left(1 - \frac{1}{(1 + r)^4}\right) = \frac{1,200}{0.1} \left(1 - \frac{1}{1.1^4}\right) = 1,200 \times 5.3349 = 6,401.88$$

The present value is larger than the price, so you should buy it.

4.30

This annuity pays you 20 times. The present value of the annuity at date 2 is

$$PV_2 = \frac{C}{r} \left(1 - \frac{1}{(1 + r)^2}\right) = \frac{2,000}{0.08} \left(1 - \frac{1}{1.08^{20}}\right) = 2,000 \times 9.8181 = 19,636.2$$
The value of the annuity at today is

\[ PV_0 = PV_2 \times \frac{1}{(1 + r)^2} = \frac{19,636.2}{1.08^2} = 16,834.88 \]

4.33

\[ \text{a.} \]

The value of the saving 5 years from now must be equal to $25,000 to finance the car. Then the present value of the saving must be equal to the present value of the car today.

\[ PV_0 (\text{car}) = PV_0 (\text{saving}) \]

\[ \frac{25,000}{1.07^5} = C \times (1 - \frac{1}{1.07^5}) \]

\[ 17,824.65 = C \times 4.1002 \]

\[ C = 4,347.265 \]

So you need to save $4,347.265 every year to finance the car.

\[ \text{b.} \]

You need to deposit M so that its future value at the end of year 5 is $25,000. Then

\[ M \times 1.07^5 = 25,000 \]

\[ M = 17,824.5 \]

4.36

You will make deposits for 15 times at date 1 through date 15. The tuition for the first child will be paid at date 16 though 19. The tuition for the second child will be paid at date 18 through date 21. You have to save enough so that

\[ PV_0 (\text{saving}) = PV_0 (\text{tuition1}) + PV_0 (\text{tuition2}) \]

\[ X \times A_{0.15}^{15} = \frac{21,000 \times A_{0.15}^4}{1.15^{15}} + \frac{21,000 \times A_{0.15}^4}{1.15^{17}} \]

\[ X \times \frac{1}{r} \left[ 1 - \frac{1}{(1 + r)^{15}} \right] = \frac{21,000 \times \frac{1}{r} \left[ 1 - \frac{1}{(1 + r)^4} \right]}{(1 + r)^{15}} + \frac{21,000 \times \frac{1}{r} \left[ 1 - \frac{1}{(1 + r)^4} \right]}{(1 + r)^{17}} \]
5.8474 \times X = \frac{\$21,000 \times 2.8550}{1.15^{15}} + \frac{\$21,000 \times 2.8550}{1.15^{17}}
X = \$2,212.87.

2. Reviewing indifference curves

a. Define and plot in a graph this individual’s budget constraint.

The budget constraint can be defined either at time \( t = 0 \)

\[
\frac{w_1}{1 + r} + w_0 = \frac{c_1}{1 + r} + c_0,
\]

that is,

\[
\frac{100}{1.07} + 50 = 143.46 = \frac{c_1}{1.07} + c_0,
\]

or at time \( t = 1 \)

\[
w_1 + w_0(1 + r) = c_1 + c_0(1 + r),
\]

that is,

\[
100 + 50(1.07) = 153.5 = c_1 + c_0(1.07),
\]

and both ways are equivalent.

If we plot it, we obtain
b. For each of the following utility functions, compute and plot in a graph the optimal consumption choice of the individual:

1. \( u(c_0, c_1) = \min\{c_0, c_1\} \);

The individual’s problem is

\[
\max_{c_0, c_1} u(c_0, c_1) = \min\{c_0, c_1\}
\]

\[
s.to \quad 143.46 = \frac{c_1}{1.07} + c_0
\]

Since these goods are complements in the proportion 1 to 1, the solution for this problem is \( c_0 = c_1 \), then

\[
143.46 = \frac{c_1}{1.07} + c_1 = \frac{c_1(2.07)}{1.07},
\]

and solving for

\[
c_1 = c_0 = 74.15.
\]

If we plot it, we have
2. \( u(c_0, c_1) = c_0c_1; \)

The individual's problem is

\[
\max_{c_0, c_1} u(c_0, c_1) = c_0c_1 \\
\text{s.t. } 143.46 = \frac{c_1}{1.07} + c_0
\]

The solution for this problem can be obtain either by setting up the Lagrangian or solving for \( c_0 \) or \( c_1 \) in the budget constraint and plugging it into the utility function. Using the Lagrangian we obtain the following first order conditions

\[
\begin{align*}
[c_0] & \rightarrow c_1 - \lambda = 0, \\
[c_1] & \rightarrow c_0 - \frac{\lambda}{1.07} = 0, \\
[\lambda] & \rightarrow 143.46 - \frac{c_1}{1.07} - c_0 = 0,
\end{align*}
\]

where \( \lambda \) is the Lagrange multiplier associated to the constraint. Solving for \( c_0 \) and \( c_1 \) we obtain

\[
\begin{align*}
c_0^* &= 71.73, \\
c_1^* &= 76.75.
\end{align*}
\]

If we plot it, we have
3. \( u(c_0, c_1) = c_0 + c_1 \).

The individual’s problem is

\[
\max_{c_0, c_1} u(c_0, c_1) = c_0 + c_1 \\
\text{s.t.} \quad 143.46 = \frac{c_1}{1.07} + c_0
\]

The solution for this problem can be obtain either by setting up the Lagrangian or solving for \( c_0 \) or \( c_1 \) in the budget constraint and plugging it into the utility function. Using the Lagrangian we obtain the following first order conditions

\[
\begin{align*}
[c_0] & \quad \rightarrow \quad 1 - \lambda = 0, \\
[c_1] & \quad \rightarrow \quad 1 - \frac{\lambda}{1.07} = 0, \\
[\lambda] & \quad \rightarrow \quad 143.46 - \frac{c_1}{1.07} - c_0 = 0,
\end{align*}
\]

where \( \lambda \) is the Lagrange multiplier associated to the constraint. Notice that in this case, \( 1.07 \neq 1 \),

so the solution is a corner solution. Given that these goods are perfect substitutes, the individual is indifferent between consuming \( c_0 \) or \( c_1 \), so he will choose the less expensive good. That is,

\[
c_0^* = 0, \\
c_1^* = 153.5
\]

Graphically, there will be no tangency point,
c. Assume instead, that the individual obtains the following investment opportunity: he can invest $20 at time $t = 0$ at an 8% interest rate. How does the optimal consumption choice of this individual change for each of the preferences given above? Is he better or worse off than before? Explain.

In all cases, the budget constraint of the individuals will shift outwards, due to the new investment. At $t = 0$ the constraint of the individual will be

$$w_0 = c_0 + i_0,$$

that is, his labor income is devoted now not only to consumption but to investment, so

$$50 = c_0 + 20,$$

$$50 - 20 = 30 = c_0.$$

At time $t = 1$ he will obtain $20 \times (1 + 0.08) = $21.6, that is

$$w_1 + i_1 = c_1,$$

$$100 + 21.6 + (30 - c_0)(1 + 0.07) = c_1,$$

so the new budget constraint at $t = 1$ is

$$153.7 = c_1 + 1.07c_0.$$

In all the three cases above this new budget constraint will enlarge the consumption possibilities of the individual, and both consumption today and tomorrow will increase.

1. The new values for consumption are: $c_0^* = c_1^* = 76.85$.
2. The new consumption values are: $c_0^* = 71.82$, and $c_1^* = 76.85$.
3. In this case, we obtain again a corner solution, and the result is $c_0^* = 0$, and $c_1^* = 153.7$. 
3. More on net present values...

1. Assume that the real interest rate per month is $r = 4\%$. Should you undertake the project?

We have to compute the net present value for the investment. We will value it in April since it is when the decision needs to be made.

\[
\begin{array}{cccc}
\text{-$200} & \text{-$200} \\
\text{-$800} & \text{-$800} \\
\text{-$3,200} & \text{-$200} & +$1,560 & \ldots & +$1,560 \\
\end{array}
\]

\[
\begin{align*}
4/06 & \quad 8/06 & \quad 9/06 & \quad 8/07 \\
\end{align*}
\]

\[r = 0.04\]

We compute the NPV by parts. First, since September 06 we have a total of $1,000$ of costs and $1,560$ of revenues. Then in September 06 the NPV of this flow is

\[
A_{9/06} = $1,560 \left[ 1 + \ldots + \frac{1}{(1 + r)^{11}} \right] - $1,000 \left[ 1 + \ldots + \frac{1}{(1 + r)^{11}} \right],
\]

that is,

\[
NPV(9/06) = (\$1,560 - \$1,000) \left[ 1 + \ldots + \frac{1}{(1 + 0.04)^{11}} \right] =
\]

applying the formula, we have

\[
= (\$1,560 - \$1,000) \left[ \frac{1.04}{0.04} - \frac{1}{1.04^{11} 0.04} \right] =
\]

\[
= (\$1,560 - \$1,000) \frac{1.04}{0.04} \left[ 1 - \frac{1}{1.04^{11}} \right] = $5,465.87
\]

this amount in April 06 is

\[
A_{4/06} = \frac{A_{9/06}}{(1 + r)^5} = \frac{$5,465.87}{1.04^5} = $4,492.55.
\]

Next, we need to account for the costs before September 06. This is a single payment in August 06 of $200, and an initial payment in April 06 of $3,200, that is, the final NPV is

\[
NPV = -\$3,200 - \frac{\$200}{1.04^4} + $4,492.55 = $1121.59.
\]

Since the NPV is positive, yes, you should undertake the project.
2. Now we are in December 06, and what we paid we paid, so if an investor wants to buy our business we should compute how much we would stop earning to see the selling price.

Then if we continued the business until August 07, we would have costs and revenues. The net present value would be

$$NPV = (\$1,560 - \$1,000) \left( \frac{1}{0.04} \right) \left[ 1 - \frac{1}{1.04^8} \right] = \$3,770.34.$$ 

So at least, the price we should ask for is $\$3,770.34.

(Notice: we are not computing the selling price to break even, that would be another exercise!!)

3. Now, we are still in December 06, but the conditions are different, we assume now that our earnings have been growing and expect them to keep on growing henceforth. So the selling price could be different. Let’s compute again the NPV under such circumstances

$$NPV = -\$1,000 \left( \frac{1}{0.04} \right) \left[ 1 - \frac{1}{1.04^8} \right] + earnings,$$

notice that costs are the same, they do not grow.

Regarding earnings, let’s see how they evolve. At December 06, the expected flow of earnings until August 07 is represented by

$$\frac{\$1,655.48(1 + 0.02)}{(1 + 0.04)} + \frac{\$1,655.48(1 + 0.02)^2}{(1 + 0.04)^2} + \ldots + \frac{\$1,655.48(1 + 0.02)^8}{(1 + 0.04)^8} =$$

$$= \$1,655.48 \left[ \frac{1.02}{1.04} + \left( \frac{1.02}{1.04} \right)^2 + \ldots + \left( \frac{1.02}{1.04} \right)^8 \right] =$$

$$= \frac{\$1,655.48(1.02)}{1.04} \left[ \frac{1}{1 - \left( \frac{1.02}{1.04} \right)} - \left( \frac{1.02}{1.04} \right)^8 \frac{1}{1 - \left( \frac{1.02}{1.04} \right)} \right] =$$

$$= \$1,655.48 \left( \frac{1.02}{1.04} \right) \left[ \frac{1.04}{1.04 - 1.02} - \left( \frac{1.02}{1.04} \right)^8 \frac{1.04}{1.04 - 1.02} \right] =$$

$$= \$1,655.48 \left( \frac{1.02}{1.04} \right) \left[ \frac{1.04}{0.02} - \left( \frac{1.02}{1.04} \right)^8 \frac{1.04}{0.02} \right] =$$

$$= \$1,655.48 \left( \frac{1.02}{1.04} \right) \left[ 1 - \left( \frac{1.02}{1.04} \right)^8 \right] = \$12,147.71$$

Then, the NPV will be

$$NPV = -\$1,000 \left( \frac{1}{0.04} \right) \left[ 1 - \frac{1}{1.04^8} \right] + earnings =$$

$$= -\$6,732.74 + $12,147.71 = $5,414.97.$$ 

So the selling price in this case is $\$5,414.97.
4. Life annuities.

1. The flows of cash is as follows:

\[-$25,000 \quad -$25,000 \quad C \quad \cdots \quad C\]

\[
\begin{array}{cccc}
50 & 64 & 65 & 92 \\
\end{array}
\]

\[r = 0.025\]

Then, the amount \(C\) Annie would receive during her retirement will be

\[
$25,000 \left[ 1.025^{14} + \ldots + 1 \right] = \frac{C}{0.025} \left[ 1 - \frac{1}{1.025^{28}} \right],
\]

where 28 is the number of years she is expected to live after retirement. Then we obtain \(C = 22,454.33\) per year.

2. Now, the only change will be WHEN she starts putting money into the investment, then

\[
$25,000 \left[ 1.025^4 + \ldots + 1 \right] = \frac{C}{0.025} \left[ 1 - \frac{1}{1.025^{28}} \right],
\]

which yields \(C = 6,581.97\) per year.

Differences:

- less time to add to the investment;
- same expected retirement period.