SOLUTIONS PROBLEM SET 5

5A.1 a. The present value of any coupon bond is the present value of its coupon payments and face value. Match each cash flow with the appropriate spot rate. For the cash flow that occurs at the end of the first year, use the one-year spot rate. For the cash flow that occurs at the end of the second year, use the two-year spot rate.

\[ P = \frac{C_1}{(1+r_1)} + \frac{(C_2+F)}{(1+r_2)^2} \]
\[ = \frac{60}{1.1} + \frac{(60+1000)}{(1.11)^2} \]
\[ = 54.55 + 860.32 \]
\[ = 914.87 \]

The price of the bond is $914.87.

b. The yield to the maturity is the discount rate, \( y \), which sets the cash flows equal to the price of the bond.

\[ P = \frac{C_1}{(1+y)} + \frac{(C_2+F)}{(1+y)^2} \]
\[ \frac{914.87}{(1.09)} = \frac{60}{(1+y)} + \frac{(60+1000)}{(1+y)^2} \]
\[ y = .1097 = 10.97\% \]

The yield to maturity is 10.97%.

5A.3 Apply the forward rate formula to calculate the one-year rate over the second year.

\[ (1+r_1) \times (1+f_2) = (1+r_2)^2 \]
\[ (1.09) \times (1+f_2) = (1.10)^2 \]
\[ f_2 = .1101 \]

The one-year forward rate over the second year is 11.01%.

5A.4 Calculate the forward rate over each year.

a. Apply the forward rate formula to calculate the one-year forward rate over the second year.

\[ (1+r_1) \times (1+f_2) = (1+r_2)^2 \]
\[ (1.05) \times (1+f_2) = (1.07)^2 \]
\[ f_2 = .0904 = 9.04\% \]

The one-year forward rate over the second year is 9.04%.

b. Apply the forward rate formula to calculate the one-year forward rate over the third year.

\[ (1+r_2)^2 \times (1+f_3) = (1+r_3)^3 \]
\[ (1.07) \times (1+f_3) = (1.10)^3 \]
\[ f_3 = .1625 = 16.25\% \]

The one-year forward rate over the third year is 16.25%.
2. (a) $r_1^0$?

\[ \frac{952.38}{1 + r_1^0} = 1,000 \rightarrow r_1^0 = 5\% \]

(b) $r_2^0$?

\[ \frac{898.45}{1 + r_2^0} = 1,000 \rightarrow r_2^0 = 5.5\% \]

(c) $r^f$

\[ (1 + r_2^0)^2 = (1 + r_1^0)(1 + r^f) \rightarrow r^f = 6\% \]

i.

\[ P = \frac{100}{1.0525} + \frac{1,100}{(1.0525)^2} = 1,088.01 \]

ii. Maximum price?

\[ P_{\text{max}} = \frac{100}{1 + r_1^0} + \frac{1,100}{(1 + r_0^0)^2} = \frac{100}{1.05} + \frac{1,100}{(1.055)^2} = 1,083.54 \]

iii. Mistake in calculating the yield? $P_{\text{max}} = 1,083.54 < 1,088.01 = P$, so the yield should be higher (recall the negative relationship between price and yield)

iv.

\[ \frac{1,083.54}{1 + y} + \frac{1,100}{(1 + y)^2}, \]

that is,

\[ 1,083.54(1+y)^2 = 100(1+y) + 1,100, \]

\[ 1,083.54(1+y)^2 - 100(1+y) - 1,100 = 0, \]

\[ 1,083.54x^2 - 100x - 1,100 = 0, \]

\[ x = \frac{-(100) \pm \sqrt{(-100)^2 - 4 \times 1,083.54 \times (-1,100)}}{2 \times 1,083.54} = 1.0548 \]

we disregard the negative root. Then, $y = 5.48\%$. And in fact, it is higher than the one given by the broker.

i. $E(\tilde{r}_1) = 0.06 \times 0.5 + 0.55 \times 0.3 + 0.05 \times 0.2 = 0.0565 \rightarrow 5.65\%$.

ii. Notice that $E(\tilde{r}_1) = 5.65\% \neq 6\% = r^f$

iii. $r^f > E(\tilde{r}_1)$, so yes.

3.

\[ P = \frac{80}{1.08} + \frac{80}{1.08 \times 1.1} + \frac{1,080}{1.12 \times 1.1 \times 1.08} = 953.10, \]

\[ 953.10 = \frac{80}{1 + y} + \frac{80}{(1 + y)^2} + \frac{1,080}{(1 + y)^3}, \]

\[ 953.10(1 + y)^3 - 80(1+y)^2 - 80(1+y) - 1,080 = 0, \]

from here we obtain $y = 9.88\%$. 

4. (a) 
\[ P = \frac{9}{1.07} + \frac{109}{(1.08)^2} = 101.87, \]
be careful to match each cash flow with the appropriate spot rate!!!

(b) 
\[ 101.87 = \frac{9}{1 + y} + \frac{109}{(1 + y)^2}, \]
\[ 101.87(1 + y)^2 - 9(1 + y) - 109 = 0, \]
\[ 101.87x^2 - 9x - 109 = 0, \]
\[ x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4 \times 101.87 \times (-109)}}{2 \times 101.87} = 1.0795, \]
that is, \( y = 7.95\% \).

(c) 
\[ (1 + r_2^0)^2 = (1 + r_1^0)(1 + r^f), \]
\[ (1 + 0.08)^2 = (1 + 0.07)(1 + r^f) \rightarrow r^f = 9\%, \]
and \( E(\tilde{r}_1^1) = r^f = 9\% \)

(d) \( r^f = E(\tilde{r}_1^1) + 1\% = 10\% \).

5. (a) \( r_1^0 \rightarrow \)
\[ r_1^0 = \frac{1,000}{942.8} - 1 = 6.07\% \]

(b) \( r_4^0 \rightarrow \)
\[ r_4^0 = \left( \frac{1,000}{820.9} \right)^{\frac{1}{4}} - 1 = 5.06\% \]

(c) \( r_{10}^0 \rightarrow \)
\[ r_{10}^0 = \left( \frac{1,000}{600.4} \right)^{\frac{1}{10}} - 1 = 5.23\% \]

(d) It is a downward sloping yield curve a the beginning but the slopes upward.