Lesson 1.
Net Present Value

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1. **Introduction**

When deciding to invest or not, a firm or an individual has to decide what to do with the money today.

So we need to compare money today with money in the future.

What’s the relationship between $1 today and $1 tomorrow?

Is it worth the same $1 today as $1 tomorrow/yesterday?

\[
\begin{array}{ccc}
\text{time } t & \text{time } t+1 & \$1? \\
$1 & \rightarrow & \text{more?} \\
\text{} & & \text{less?}
\end{array}
\]

This is what is called the “time-value-of-money” concept.
2. The one-period case

Three equivalent concepts: future value, present value and net present value.

**Example (page 61 in book):** a financial analyst at a leading real estate firm is thinking about recommending that Kaufman & Broad invest in a piece of land that costs $85,000. She is certain that next year the land will be worth $91,000, a sure $6,000 gain. Given that the guaranteed interest rate in the bank is 10%, should Kaufman & Broad undertake the investment in land?
2.1 **Future value (or compound value):** the value of a sum after investing over one or more periods.

If the money is invested in the bank, next year they would have

\[
85,000 \times (1 + 0.1) = 93,500
\]

since

future value $93,500 > 91,000

then

**Invest everything in the bank.**

The general formula is

\[
FV = C_0 \times (1 + r)
\]  \hspace{1cm} (1)

where

- \(C_0\) is cash flow today (at time 0),

- and \(r\) is the appropriate interest rate.
2.2 Present value: the amount of money that should be put in the alternative investment (the bank) in order to obtain the expected amount next year.

In our example, present value is,

\[ PV \times (1 + 0.1) = $91,000 \]

solving for \( PV \)

\[ PV = \frac{$91,000}{1.1} = $82,727.27 \]

since

present value $82,727.27 < $85,000

then

Do not to buy the land.
The general formula is

$$PV = \frac{C_1}{1+r}$$

(2)

where

- $C_1$ is cash flow at date 1,
- and $r$ is the appropriate interest rate (the one required to buy the land) also called \textit{discount rate}.
2.3 **Net present value:** the present value of future cash flows minus the present value of the cost of the investment.

The formula is

\[
NPV = PV - Cost.
\]  

(3)

In our case, we would have

\[
NPV = \frac{\$91,000}{1.1} - \$85,000 = -\$2,273
\]

Because

\[
NPV < 0
\]

**Purchase of the land should not be recommended.**

NOTE: all three methods reach the same conclusions.
3. The multiperiod case

3.1 Future value and compounding

General formula

\[ FV = C_0 \times (1 + r)^T \]  

(4)

where

- \( C_0 \) is cash flow at date 0,
- \( r \) is the appropriate interest rate, and
- \( T \) is the number of periods over which the cash is invested.

Example: $1.10 of dividend, expected to grow at a 40\% per year over the next 2 years, then in 2 years the dividend will be

\[ FV = C_0 \times (1 + r)^T = 1.10 \times (1.40)^2 = $2.156 \]
1. Net Present Value

Notice that

\[(1 + 0.40)^2 = 1 + 0.40^2 + 2 \times 0.40 \quad (5)\]

**Compounding:** the process of leaving money in the capital market and lending it for another year.

Notice that

\[2.156 > 1.10 + 2 \times [1.10 \times 0.40] = 1.98\]

**Simple interest:** interest payments are not reinvested, i.e. \(T \times r\). In our example,

\[2 \times 0.40 = 0.80\]

**Compound interest:** each interest payment is reinvested, i.e. \(r^T\). In our example,

\[0.40^2 = 0.16\]

then

\[2.156 = 1.10 \times (1 + 0.16 + 0.80)\]
What rate is enough?  Alternatively, we may not know the interest rate. Example:

Assume the total cost of a college education will be $50,000 when your child enters college in 12 years. You have $5,000 to invest today. What rate of interest must you earn on your investment to cover the cost of your child’s education?

\[ FV = C_0(1 + r)^T \]

\[ $50,000 = $5,000(1 + r)^{12} \]

\[ \frac{$50,000}{$5,000} = (1 + r)^{12} \]

\[ r = \left( \frac{50,000}{5,000} \right)^{\frac{1}{12}} - 1 = 0.2115 \]
3.2 Present value and compounding

**Discounting:** the process of calculating the present value of a future cash flow. It is the opposite of compounding.

Example: how much would an investor have to set aside today in order to have $20,000 five years from now if the current rate is 15%?

\[ PV(1 + r)^T = C_T \rightarrow PV = \frac{C_T}{(1 + r)^T} \]

\[ PV(1 + 0.15)^5 = $20,000 \]

\[ PV = \frac{$20,000}{(1.15)^5} = $9,943.53 \]

**Present value factor:** the factor used to calculate the present value of a future cash flow.

\[ \frac{1}{(1 + r)^T}. \]
3.3 Multiperiod net present value

NPV of a $T$—period project can be written as

\[ NPV = -C_0 + \frac{C_1}{1 + r} + \frac{C_2}{(1 + r)^2} + \ldots + \frac{C_T}{(1 + r)^T} \]

\[ NPV = -C_0 + \sum_{i=1}^{T} \frac{C_i}{(1 + r)^i}. \] (6)
4. Compounding periods

Compounding an investment $m$ times a year for $T$ years provides for future value of wealth

$$FV = C_0 \left(1 + \frac{r}{m}\right)^{mT}$$

where

- $C_0$ is the initial investment,
- $r$ is the stated annual interest rate (also called annual percentage rate),
- $m$ is how many times a year the investment is compounded, and
- $T$ is the total number of periods of the investment.
Example: if you invest $50 for 3 years at 12% compounded semiannually, your investment will grow to

\[ FV = 50 \left(1 + \frac{0.12}{2}\right)^{2\times3} = 50(1+0.06)^6 = 70.93 \]

Example: what is the end-of-year wealth of $1 invested for one year at a stated annual interest rate of 24% compounded monthly?

\[ FV = 1 \left(1 + \frac{0.24}{12}\right)^{12\times1} = 1(1+0.02)^{12} = 1.2682 \]

Example: what is the wealth at the end of 5 years of investing $5,000 at a stated annual interest rate of 12% per year, compounded quarterly?

\[ FV = 5,000 \left(1 + \frac{0.12}{4}\right)^{4\times5} = 5,000(1 + 0.03)^{20} = 9,030.50 \]
Effective annual interest rates: the annual rate that would give us the same end-of-investment wealth after $T$ years.

In the example above,

\[ 50(1 + EAR)^3 = 70.93 \]

\[ EAR = \left( \frac{70.93}{50} \right)^{\frac{1}{3}} - 1 = 0.1236 = 12.36\%. \]

So, investing at 12.36% compounded annually is the same as investing at 12% compounded semianually.

That is,

\[ (1 + r_{EAR}) = \left(1 + \frac{r}{m}\right)^m \]  \hspace{1cm} (8)

\[ r_{EAR} = \left(1 + \frac{r}{m}\right)^m - 1. \]  \hspace{1cm} (9)
Continuous Compounding

The general formula for the future value of an investment compounded continuously over many periods can be written as

\[ FV = C_0 \times e^{rT} = C_0 \times \exp(rT) \quad (10) \]

where

- \( C_0 \) is cash flow at date 0,
- \( r \) is the stated annual interest rate,
- \( T \) is the number of periods over which the cash is invested.
5. Simplifications

**Perpetuity:** a constant stream of cash flows that lasts forever.

\[
PV = \frac{C}{(1 + r)} + \frac{C}{(1 + r)^2} + \frac{C}{(1 + r)^3} + \ldots = \frac{C}{r} \quad (11)
\]

Example: the British bonds called *consols*. The present value of a consol is the present value of all of its future coupons.
Example: consider a perpetuity paying $100 a year. If the relevant interest rate is 8%, what is the value of the consol?

\[ PV = \frac{\$100}{0.08} = \$1,250 \]

What is the value of the consol if the interest rate goes down to 6%?

\[ PV = \frac{\$100}{0.06} = \$1,666.67 \]

Note: the value of the perpetuity rises with a drop in the interest rate, and vice versa.
Annuity: a stream of constant cash flows that lasts for a fixed number of periods.

\[ PV = \frac{C}{(1 + r)} + \frac{C}{(1 + r)^2} + \frac{C}{(1 + r)^3} + \cdots + \frac{C}{(1 + r)^T} \]  

(12)

\[ PV = \frac{C}{r} \left[ 1 - \frac{1}{(1 + r)^T} \right] \]  

(13)

Intuition: an annuity is valued as the difference between two perpetuities: one that starts at time 1 minus another one that starts at time \( T + 1 \).
Example: If you can afford a $400 monthly car payment, how much car can you afford if interest rates are 7% on 36-month loans?

\[ PV = \frac{400}{0.07} \left[ 1 - \frac{1}{(1 + \frac{0.07}{12})^{36}} \right] = 12,954.59 \]

Example: What is the present value of a 4-year annuity of $1000 per year that makes its first payment 2 years from today if the discount rate is 9%?

First, let’s bring all the value to the starting time of the annuity

\[ PV = 1000 + \frac{1000}{1.09} + \frac{1000}{1.09^2} + \frac{1000}{1.09^3} = 3531.29 \]

this amount today is

\[ PV = \frac{3531.29}{1.09^2} = 2972.22 \]
6. Net present value: first principles of finance

Making consumption choices over time: the intertemporal budget constraint

\[ PV(\text{consumption}) \leq PV(\text{total wealth}) \]

So, given

- Initial assets \( A_0 \)
- Annual wage \( w_t \) at any period \( t \)
- Real interest rate \( r \)
- Living from \( t = 0 \) to \( t = T \)
and individual’s consumption should be such that
\[ c_0 + \frac{c_1}{1 + r} + \ldots + \frac{c_T}{(1 + r)^T} \leq A_0 + w_0 + \frac{w_1}{1 + r} + \ldots + \frac{w_T}{(1 + r)^T} \]

Let’s consider a person who lives for two periods. This person’s budget constraint can be explained with the net present value.

Assumptions:

- no initial assets,
- income \( w_0 \) and \( w_1 \),
- \( r \) is the real interest rate in the financial market.
This can be represented

\[ c_0 + \frac{c_1}{1 + r} \leq w_0 + \frac{w_1}{1 + r} \]

if all is consumed today (i.e. \( c_1 = 0 \)), then

\[ c_0 = w_0 + \frac{w_1}{1 + r} \]

but if all is saved and consumed tomorrow (i.e. \( c_0 = 0 \)), then

\[ c_1 = w_0(1 + r) + w_1, \]

which are the intercepts to plot the budget constraint.
Changes in the interest rate change the slope of the budget constraint.

Changes in income shift the budget constraint.

**Making consumption choices over time: consumer preferences**  Individual’s preferences can be represented by utility functions.

Indifference curves (IC): combinations of goods that give the consumer the same level of happiness or satisfaction.

Some properties:

- higher IC are preferred to lower IC

- IC are downward sloping (the slope is the *marginal rate of substitution*)
- IC do not cross each other

- IC are bowed inward (convex)

Making consumption choices over time: optimal consumption choice

Individuals choose consumption intertemporally by maximizing their utility subject to the budget constraint.

For our example, the problem can be written as follows:

$$\max_{c_0, c_1} u(c_0, c_1)$$

subject to

$$c_0 + \frac{c_1}{1 + r} \leq w_0 + \frac{w_1}{1 + r}.$$
1. Net Present Value

**Illustrating the investment decision**

Consider an investor who has an initial endowment of income, $w_0$, of $40,000$ this year and $w_1 = $55,000 next year.

Suppose that he faces a 10% interest rate and is offered the following investment:

First of all, should the individual undertake the project? Compute NPV

$$NPV = -25,000 + \frac{30,000}{1.1} = 2,272.72 > 0 \rightarrow YES!$$
1. Net Present Value

**Budget constraint without investment**

\[ c_1 + c_0(1 + r) = w_0(1 + r) + w_1, \]

that is

\[ c_1 + 1.1c_0 = \$99,000 \]

**Budget constraint with investment**  Notice that if he invests at \( t = 0 \) he will have left

\[ w_0 - \$25,000 - c_0. \]

At time \( t = 1 \) he will get

\[ w_1 + \$30,000. \]

Then the budget constraint becomes

\[ c_1 + c_0(1.1) = \]

\[ = (\$40,000 - \$25,000)(1.1) + \$55,000 + \$30,000 \]

that is

\[ c_1 + 1.1c_0 = \$101,500 \]
If we plot both budget constraints we obtain

That is, the consumption possibilities frontier is expanded if we undertake this investment project, which has a NPV > 0.
Where will the consumer stay?

We need some assumption on the preferences.

Assume \( u(c_0, c_1) = c_0 c_1 \). Then the problem to solve is

\[
\max_{c_0, c_1} c_0 c_1
\]

subject to

\[
c_1 + 1.1c_0 = 99,000
\]

in the case of no investment.

Optimal solution \((A)\) is:

\[
(c^*_0, c^*_1) = (45,000, 49,500).
\]

If we consider the investment opportunity, the problem to solve is

\[
\max_{c_0, c_1} c_0 c_1
\]

subject to

\[
c_1 + 1.1c_0 = 101,500
\]
and the optimal solution \((B)\) is:

\[
(c_0^*, c_1^*) = (\$46, 136.36, \$50, 749.99).
\]

If we plot it, we have

The investment decision allows the individual to reach a higher indifference curve, and therefore, he is better off.