Review of portfolio mathematics
(from Bodie, Kane and Marcus (1999))

Consider the problem of Humanex, a nonprofit organization deriving most of its income from the return on its endowment. Years ago, the founders of Best Candy willed a large block of Best Candy stock to Humanex with the provision that Humanex may never sell it. This block of shares now comprises 50% of Humanex’s endowment. Humanex has free choice as to where to invest the remainder of its portfolio.

The value of Best Candy stock is sensitive to the price of sugar. In years when world sugar crops are low, the price of sugar rises significantly and Best Candy suffers considerably losses. We can describe the fortunes of Best Candy stock using the following scenario analysis:

<table>
<thead>
<tr>
<th>Normal Year for Sugar</th>
<th>Abnormal Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bullish Stock Market</td>
<td>Abnormal Year</td>
</tr>
<tr>
<td>Bearish Stock Market</td>
<td>Sugar Crisis</td>
</tr>
<tr>
<td>Probability</td>
<td>0.5</td>
</tr>
<tr>
<td>Rate of return</td>
<td>25%</td>
</tr>
<tr>
<td></td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>10%</td>
</tr>
<tr>
<td></td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td>-25%</td>
</tr>
</tbody>
</table>

**Rule 1** The mean or expected return of an asset is a probability-weighted average of its return in all scenarios. Calling $\Pr(s)$ the probability of scenario $s$ and $r(s)$ the return in scenario $s$, we may write the expected return, $E(r)$, as

$$E(r) = \sum_s \Pr(s)r(s)$$  \hspace{1cm} (1)

Applying this formula to the case at hand, with three possible scenarios, we find that the expected rate of return of Best Candy’s stock is

$$E(r_{\text{Best}}) = (0.5 \times 25) + (0.3 \times 10) + (0.2 \times (-25)) = 10.5\%$$

**Rule 2** The variance of an asset’s returns is the expected value of the squared deviations from the expected return. Symbolically,

$$\sigma^2 = \sum_s \Pr(s) [r(s) - E(r)]^2$$  \hspace{1cm} (2)

Therefore, in our example

$$\sigma^2_{\text{Best}} = 0.5 \times (25 - 10.5)^2 + 0.3 \times (10 - 10.5)^2 + 0.2 \times (-25 - 10.5)^2 = 357.25$$

The standard deviation of Best’s return, which is the square root of the variance, is $\sqrt{357.25} = 18.9\%$.

Humanex has 50% of its endowment in Best’s stock. To reduce the risk of overall portfolio, it could invest the remainder in T-bills, which yield a sure rate of return of 5%. To derive the return of the overall portfolio, we apply rule 3.
Rule 3 The rate of return on a portfolio is a weighted average of the rates of return of each asset comprising the portfolio, with portfolio proportions as weights. This implies that the expected rate of return on a portfolio is a weighted average of the expected rate of return on each component asset. In our example,

\[ E(r_{Humanex}) = 0.5 \times E(r_{Best}) + 0.5 \times r_{Bills} = (0.5 \times 10.5) + (0.5 \times 5) = 7.75\% \]

Rule 4 When a risky asset is combined with a risk-free asset, the portfolio standard deviation equals the risky asset’s standard deviation multiplied by the portfolio proportion invested in the risky asset. In our example,

\[ \sigma_{Humanex} = 0.5 \times \sigma_{Best} = 0.5 \times 18.9 = 9.45\% \]

Consider another firm SugarKane that offers excellent hedging potential for holders of Best stock because its return is highest precisely when Best’s return is lowest – during a sugar crisis. Consider Humanex’s portfolio when it splits its investment evenly between Best and SugarKane. The rate of return for each scenario is the simple average of the rates on Best and SugarKane because the portfolio is split evenly between the two stocks. A scenario analysis of SugarKane’s stock looks like this:

<table>
<thead>
<tr>
<th>Normal Year</th>
<th>Abnormal Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stock Market</td>
<td>Sugar Crisis</td>
</tr>
<tr>
<td>Probability</td>
<td>0.5</td>
</tr>
<tr>
<td>Rate of return</td>
<td>1%</td>
</tr>
<tr>
<td></td>
<td>-5%</td>
</tr>
<tr>
<td></td>
<td>35%</td>
</tr>
</tbody>
</table>

The expected rate of return on SugarKane’s stock is 6%, and its standard deviation is 14.73%. To quantify the hedging or diversification potential of an asset, we use the concepts of covariance and correlation. The covariance measures how much the returns on two risky assets move in tandem:

\[
Cov(r_{Best}, r_{SugarKane}) = \sum_s \Pr(s) [r_{Best}(s) - E(r_{Best})] [r_{SugarKane}(s) - E(r_{SugarKane})] \tag{3}
\]

In this example

\[
Cov(r_{Best}, r_{SugarKane}) = 0.5(25 - 10.5)(1 - 6) + 0.3(10 - 10.5)(-5 - 6) + 0.2(-25 - 10.5)(35 - 6) = -240.5
\]

The negative covariance confirms the hedging quality of SugarKane stock relative to Best Candy. SugarKane’s returns move inversely with Best’s.

An easier statistic to interpret than the covariance is the correlation coefficient, which scales the covariance to a value between -1 (perfect negative correlation) and +1 (perfect positive correlation). The correlation coefficient between two variables equals their covariance divided by the product of the standard deviations:

\[
\rho(\text{Best}, \text{SugarKane}) = \frac{Cov(r_{Best}, r_{SugarKane})}{\sigma_{Best}\sigma_{SugarKane}} \tag{4}
\]
In our example,
\[ \rho(\text{Best}, \text{SugarKane}) = \frac{-240.5}{18.9 \times 14.73} = -0.86 \]
This large negative correlation (close to -1) confirms the strong tendency of Best and SugarKane stocks to move inversely, or “out of phase” with one another.

**Rule 5** When two risky assets with variances \( \sigma_1^2 \) and \( \sigma_2^2 \), respectively, are combined into a portfolio with portfolio weights \( w_1 \) and \( w_2 \), respectively, the portfolio variance \( \sigma_p^2 \) is given by
\[
\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1w_2 \text{Cov}(r_1, r_2) \tag{5}
\]
In our example,
\[
\sigma_p^2 = (0.5^2 \times 18.9^2) + (0.5^2 \times 14.73^2) + 2 \times 0.5 \times 0.5 \times (-240.5) = 23.3,
\]
so that \( \sigma_p = \sqrt{23.3} = 4.83\% \).

Rule 5 for portfolio variance highlights the effect of covariance on portfolio risk. A positive covariance increases portfolio variance, and a negative covariance acts to reduce portfolio variance. This makes sense because returns on negatively correlated assets tend to be offsetting, which stabilizes portfolio returns.