Consumption and Savings

Master en Economía Internacional

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Objectives

There are important differences between the short- and the long-run relationship between aggregate consumption and income.

- Over long time horizons, (aggregate) consumption and income are almost perfectly correlated.
- At business cycle frequencies this one-to-one relationship is broken. In particular, consumption tends to fluctuate less than income as the agents use their savings to smooth consumption.

The main objective of this lecture is to consider standard models of inter-temporal consumption and savings decisions that help to understand these differences.
Outline

1. Partial equilibrium in two periods
2. General equilibrium in two periods with competitive credit markets
3. Extensions
   - $N$ periods
   - Uncertainty and risk aversion
   - Borrowing constraints
4. Testable predictions
   - Permanent income hypothesis
   - Random walk hypothesis
   - Life-cycle hypothesis
Savings motives

Why do people save?

1. Consumption smoothing - agents dislike fluctuations in consumption spending.
2. Precautionary motives — fear of unemployment etc.
3. To finance consumption during retirement
4. Purchase of real estate or durable consumption goods.

For the moment we focus on 1. Saving for retirement is treated at the end of the course, while precautionary savings is left for later.
Basic assumptions

- We consider an economy that lasts for two periods, \( t = 1, 2 \).
- In both periods, each agent \( j \in J \) receives a known endowment \( y_{j,t} \geq 0 \).
- Agents have access to a perfectly competitive bond markets.
- Individual agents take the real interest rate \( R \) as given.
- We assume complete information and we impose the condition that agents have to honor their debts.
Bonds

- A bond is a piece of paper with a promise of a future payment of a certain amount of goods to the holder of the bond.
- Bonds are issued by borrowers and handed over to lenders.
- All bonds are supposed to expire in one period.
- We denote the (real) interest rate by $R$.
- A household that buys $b_{j,t}$ in $t$ will receive $(1 + R_t) b_{j,t}$ units of the good in period $t + 1$.
- We assume complete information. All loans are repaid.
The one-period budget constraint of a representative household can be written as

\[ c_t + b_t = y_t + (1 + R)b_{t-1} \]

When \( b_{t-1} > 0 \), the resources of the household exceed the value of income

\[ y_t + (1 + R)b_{t-1} > y_t \]

By contrast, when \( b_{t-1} < 0 \), the household has to devote part of this period’s resources to pay back the loan plus interest

\[ y_t + (1 + R)b_{t-1} < y_t \]
In a two-period setting, a representative household faces the following pair of budget constraints:

\[ c_1 + b_1 = y_1, \]
\[ c_2 + b_2 = y_2 + (1 + R)b_1. \]

Note that savings in the second period are necessarily equal to zero. The household would like to borrow an infinite amount, but there are no lenders! Hence,

\[ c_2 = y_2 + (1 + R)b_1 \]
Inter-temporal budget constraint

The two budget constraints can be consolidated into a single lifetime budget constraint.

Note that:

\[ b_1 = y_1 - c_1, \]

Using this expression we can write the second-period budget constraint as

\[ c_2 = y_2 + (1 + R)(y_1 - c_1). \]

Dividing both sides by \((1 + R)\) yields the following expression:

\[
\begin{align*}
\underbrace{c_1 + \frac{c_2}{1 + R}} & \quad = \quad \underbrace{y_1 + \frac{y_2}{1 + R}} \\
\text{Present Value of Lifetime Consumption} & \quad = \quad \text{Present Value of Lifetime Income}
\end{align*}
\]

\[ y_1 + \frac{y_2}{1 + R} = x. \]
The inter-temporal budget constraint

\[ c_1 + \frac{c_2}{1+R} = x \]

\[ x(1+R) \]

Consumption and Savings Decisions

Fall 2014
Preferences

- We use the following standard specification of time-separable preferences:
  \[ U(c_1, c_2) = u(c_1) + \beta u(c_2), \]

- The same function \( u(.) \) defines utility in both periods, but the utility from future consumption is discounted at rate \( \beta \leq 1 \).
- The discount factor is commonly expressed as \( \beta = \frac{1}{1+\rho} \), where \( \rho \) represents the rate of time preference.
- Finally, \( u'(.) = \frac{\partial u(.)}{\partial c_t} > 0 \) and \( u''(.) = \frac{\partial u'(.)}{\partial c_t} < 0 \)
- In many examples we will consider \( u(c_t) = \log(c_t) \) with \( \lim_{c_t \to 0} u'(c_t) = \infty \). This avoids corner solutions.
Inter-temporal marginal rate of rate of substitution (IMRS)

The indifference curves in this two-period setting are pairs of current and future consumption that offer the same level of utility $u^0$:

$$U(c_1, c_2) = u(c_1) + \beta u(c_2) = u^0$$

Along the indifference curve

$$u'(c_1) dc_1 + \beta u'(c_2) dc_2 = 0$$

And the IMRS is defined as:

$$\frac{dc_2}{dc_1} = -\frac{u'(c_1)}{\beta u'(c_2)}$$
Indifference curves

Consumo mañana

$c_2$

$u_2$

$u_1$

Consumo
Optimization problem

The most general statement of the agent’s optimization problem is:

$$\max_{\{c_1, b_1, c_2\}} U(c_1, c_2) = u(c_1) + \beta u(c_2)$$

s.t.

\[
\begin{align*}
  c_1 + b_1 &= y_1 \\
  c_2 + b_2 &= y_2 + (1 + R)b_1 \\
  c_1 &\geq 0 \\
  c_2 &\geq 0
\end{align*}
\]
The Optimization Problem

In compact form the optimization problem can be written as

$$\max_{c_1, c_2} [u(c_1) + \beta u(c_2)]$$  \hspace{1cm} (1)

s.t.

$$\frac{c_2}{1 + R} + c_1 = x.$$  \hspace{1cm} (2)

Substituting the budget constraint into the objective function we obtain:

$$\max_{c_1} [u(c_1) + \beta u((x - c_1)(1 + R))]$$  \hspace{1cm} (3)

Note: we rule out corner solutions and we impose equality in (2).
The solution

The FOC associated with our optimization problem

$$\max_{c_1} \left[ u(c_1) + \beta u((x - c_1)(1 + R)) \right]$$

is given by:

$$u'(c_1) + \beta u'(\underbrace{(x - c_1)(1 + R)}_{c_2})(-1)(1 + R) = 0.$$ 

Or alternatively,

$$\underbrace{u'(c_1)}_{MC} = \underbrace{u'(c_2)\beta(1 + R)}_{MB} \quad \text{and} \quad \frac{u'(c_1)}{\beta u'(c_2)} = 1 + R.$$ 

$$\underbrace{\text{IMRS}}_{\text{MC}} = \underbrace{\text{MB}}_{\text{IMRS}}$$
Graphical representation of solution

\[ x(1+R) \]

\[ c_1 + \frac{c_2}{1+R} = x \]

Consumo mañana

\[ c_2^* \]

\[ c_1^* \]
The optimality condition — known as the Consumption Euler equation —

\[
\frac{u'(c_1)}{\beta u'(c_2)} = (1 + R)
\]

is nothing else than the intertemporal variant of a well-known result in consumption theory.

**IMRS = price ratio of consumption in both periods**
Slope of the optimal consumption profile

According to the Euler equation

\[ u'(c_1) = \beta(1 + R)u'(c_2) \]

there are two opposing forces that affect the inter-temporal choices of the agent

- Time preference
- The real interest rate
Perfect consumption smoothing

When \( \beta(1 + R) = 1 \), the Euler eqn. reduces to:

\[
u'(c_1) = u'(c_2).
\]

Given the strict concavity of \( u(.) \), this implies \( c_1 = c_2 \).

**Intuition:** Recall that \( \beta = 1/(1 + \rho) \). Hence, the necessary condition for perfect consumption smoothing can be written as:

\[
\beta(1 + R) = \frac{1 + R}{1 + \rho} = 1,
\]

which is only satisfied if

\[
R = \rho.
\]
Slope of consumption profiles

Using the same line of argument, one can easily demonstrate that the consumption Euler eqn

\[
\frac{u'(c_1)}{\beta u'(c_2)} = (1 + R)
\]

implies the following three results

\[
\beta(1 + R) < 1 \implies c_1 > c_2
\]
\[
\beta(1 + R) = 1 \implies c_1 = c_2
\]
\[
\beta(1 + R) > 1 \implies c_1 < c_2
\]

The above conditions play a central role in any dynamic macro model with endogenous consumption choices.
Example: logarithmic utility

Suppose that the agent’s preferences satisfy

\[ u(c_1) + \beta u(c_2) = \log(c_1) + \beta \log(c_2). \]

In this case, we can write the maximand as

\[ \max_{c_1} [\log(c_1) + \beta \log((x - c_1)(1 + R))] \]

and the associated FOC is:

\[ \frac{1}{c_1} + \beta \frac{1}{c_2}(-1)(1 + R) = 0 \]

\[ \frac{1}{c_1} = \frac{1}{c_2} \beta(1 + R) \implies c_2 = \beta(1 + R)c_1 \]

\[ \frac{c_2}{c_1} = \beta(1 + R). \]
Log Utility

Substituting \( c_2 = \beta(1 + R)c_1 \) into the lifetime budget constraint we get:

\[
c_1 + \frac{c_1 \beta(1 + R)}{1 + R} = x,
\]

And so,

\[
(1 + \beta)c_1 = x \implies c_1 = \frac{x}{1 + \beta}.
\]

With logarithmic utility the agents consume a constant fraction \( 1/(1 + \beta) \) of their lifetime resources in the first period

\[
c_2 = c_1 \beta(1 + R) = \frac{\beta}{1 + \beta}x(1 + R).
\]

\[
b_1 = y_1 - c_1 = y_1 - \frac{x}{1 + \beta} = \frac{1}{1 + \beta} \left[ \beta y_1 - \frac{y_2}{1 + R} \right]
\]
Lagrange Method

\[ L(c_1, c_2, \lambda) = u(c_1) + \beta u(c_2) + \lambda \left[ x - \frac{c_2}{1 + R} - c_1 \right] \]

The FOCs are given by:

\[ \frac{\partial L(c_1, c_2, \lambda)}{\partial c_1} = u'(c_1) - \lambda = 0 \]

\[ \frac{\partial L(c_1, c_2, \lambda)}{\partial c_2} = \beta u'(c_2) + \lambda \left( -\frac{1}{1 + R} \right) = 0 \]

\[ \frac{\partial L(c_1, c_2, \lambda)}{\partial \lambda} = x - \frac{c_2}{1 + R} - c_1 = 0 \]

- Using the fact that \( \lambda = u'(c_1) \), we obtain the same solution as before:

\[ u'(c_1) = \frac{u'(c_2) \beta (1 + R)}{MC} = \frac{u'(c_2) \beta (1 + R)}{MB} \]
Comparative statics

Let’s start with the case of an increase in $x$:

- A rise in $x$ represents a pure income effect that leads to an increase in current and future consumption;
- The effect on borrowing or lending depends on the timing of the rise in income;
- Transitory and permanent changes produce different effects;
- Any changes in $y_1$ and $y_2$ that leave the value of $x$ unchanged do not produce changes in the optimal consumption levels.
Income effects

Consumo mañana

$c_2$ $u_2$
$c_2'$ $u_1$
$c_1$ $c_1'$ $x$

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The borrowing/lending position

The changes in \((c_1^*, c_2^*)\) depend solely on the size of \(\Delta x\). By contrast, the changes in \(b_1^*\) depend critically on the timing of the income changes.

- To be more specific, with logarithmic utility,

\[
b_1^* = \frac{\beta y_1 - \frac{y_2}{1+R}}{1 + \beta}.
\]

and so

\[
\frac{\partial b_1}{\partial y_1} = \frac{\beta}{1 + \beta} \implies 0 \quad y \quad \frac{\partial b_1}{\partial y_2} = -\frac{1}{(1 + \beta)(1 + R)} = < 0.
\]

- After an increase in \(y_1\) the agent chooses a higher \(b_1^*\). That is, lenders will lend more, while borrowers will borrow less. Similarly, after an increase in \(y_2\) the agents reduce the value of \(b_1^*\).
Interest rate changes

An increase in the interest rate provokes both an income and a substitution effect:

- **substitution effect:** The higher interest rate lowers the cost of consumption in the second period and makes saving more attractive.

- **Income effect:** For given values of $b_1$, the rise in $R$ reduces the feasible consumption levels for borrowers while it raises the wealth of lenders. The former need to pay higher interest payments, while the latter receive higher interest payments.
Higher interest rates

First-period borrowers

\[ D_I = \text{dotación inicial} \]

\[ c_1 + \frac{c_2}{1+R} = x \]
Higher interest rates

First-period lenders

\[ D_2 = \text{dotación inicial} \]

\[ c_1 + \frac{c_2}{1 + R} = x \]

\[ b_1 > 0 \]
The sum of income and substitution effects
The combined effect of a higher interest rate on the consumption levels depends on the sign of $b_1$.

**First-period lenders** ($b_1^* > 0$)
- *Income effect*: $\Delta c_1^* > 0$ & $\Delta c_2^* > 0$
- *Subst. effect*: $\Delta c_1^* < 0$ & $\Delta c_2^* > 0$
- *Combined effect*: $\Delta c_1^* \geq 0$ & $\Delta c_2^* > 0$
- $\Delta c_1^* > 0$ if the income effect dominates the substitution effect.

**First-period borrowers** ($b_1^* < 0$):
- *Income effect*: $\Delta c_1^* < 0$ & $\Delta c_2^* < 0$
- *Subst. effect*: $\Delta c_1^* < 0$ & $\Delta c_2^* > 0$
- *Combined effect*: $\Delta c_1^* < 0$ & $\Delta c_2^* \geq 0$
- $\Delta c_2^* < 0$ if income effect dominates the substitution effect.
Disentangling income and substitution effects

First-period borrowers

\[ D_1 = \text{dotación inicial} \]

A es la elección original

C es la nueva elección

Desde A hasta B, Efecto Sustitución

Desde B hasta C, Efecto Renta
Disentangling income and substitution effects

First-period lenders

\[ D_1 = \text{dotación inicial} \]

A es la elección original

\[ E \] es la nueva elección

Desde A hasta B, Efecto Sustitución

Desde B hasta E, Efecto Renta
Example log utility

With logarithmic preferences, the savings rate does NOT depend on the interest rate, because the income and substitution effect cancel out against each other.

- Recall that
  
  \[ c_1^* = \frac{x}{1 + \beta} \text{ and } c_2^* = \frac{\beta x(1 + R)}{1 + \beta}. \]

  with \( x = y_1 + \frac{y_2}{1+R} \).

- Hence, the savings rate \( c_1^*/x = \frac{1}{1+\beta} \) does not depend on \( R \). But,

  \[ \frac{\partial c_1^*}{\partial R} = -\frac{1}{1 + \beta} \frac{y_2}{(1 + R)^2} < 0 \]
  \[ \frac{\partial c_2^*}{\partial R} = \frac{\beta}{1 + \beta} y_1 > 0 \]
The elasticity of intertemporal substitution

Economists measure the willingness of agents to substitute future for current consumption with the elasticity of intertemporal substitution.

\[ EIS = \frac{\partial \ln(c_{t+1}/c_t)}{\partial \ln(\beta u'(c_{t+1})/u'(c_t))} \]

Computation

1. Derive the expression for the Intertemporal Marginal Rate of Substitution (IMRS)
2. Calculate the elasticity of the IMRS with respect to the ratio \((c_{t+1}/c_t)\).
3. The EIS is the inverse.
Example: Log Utility

For the case of logarithmic preferences $U = \log(c_1) + \beta \log(c_2)$ we have:

$$IMRS = \frac{dc_2}{dc_1} = -\frac{c_2}{\beta c_1}$$

The elasticity of the IMRS w.r.t. $c_2/c_1$ is given by

$$\frac{\partial \left(-\frac{c_2}{\beta c_1}\right)}{\partial \left(\frac{c_2}{c_1}\right)} \star \left(\frac{c_2}{c_1}\right) = -\frac{1}{\beta} \star (-\beta) = 1$$

The EIS is the inverse of this number and so is 1.
Interest-rate sensitivity of consumption growth

The EIS is of interest because it also measures the elasticity of the optimal consumption growth with respect to the interest rate.

Given that $IMRS = (1 + R)$ we have:

$$\left(\frac{d\left(\frac{c_2^*}{c_1^*}\right)\left(\frac{c_2^*}{c_1^*}\right)}{d\left(\frac{1+R}{1+R}\right)}\right) = \frac{d\left(\frac{c_2^*}{c_1^*}\right)}{d\left(1 + R\right)} \cdot \left(1 + R\right) \left(\frac{c_2^*}{c_1^*}\right)$$

The above statistic can be matched to the observed elasticity in data on consumption.
General Equilibrium

So far, we have assumed the existence of a competitive credit market. In our *pure exchange economy* borrowing and lending may occur due to

- Differences in the (life-cycle) profile of income
- Differences in time preferences

In equilibrium, the interest rate adjusts to clear the bond market, i.e.

\[ B_t = \sum_{j \in J} b_{j,t} = 0 \]

where \( J \) denotes the set of agents in the economy.
Example: no aggregate uncertainty

Suppose there are two groups of agents in the economy with different income profiles.

- $N_A$ agents of type A with $y_{A,1} = 1$ and $y_{A,2} = 0$.
- $N_B$ agents of type B with $y_{B,1} = 0$ and $y_{B,2} = 1$.
- All agents have the same strictly concave utility function

$$U(c_{i,1}, c_{i,2}) = u(c_{i,1}) + \beta u(c_{i,2})$$

Notice, when $N_A = N_B = N$ there is no aggregate uncertainty or volatility and in equilibrium all agents consume the same amount in both periods.
Logarithmic utility

With logarithmic utility each individual wishes to save an amount

\[ b_{i,1} = \frac{1}{1 + \beta} \left( \beta y_{i,1} - \frac{y_{i,2}}{1 + R} \right) \]

Hence, the equilibrium condition reduces to

\[ N_A \frac{\beta}{1 + \beta} - N_B \frac{1}{(1 + \beta)(1 + R)} = 0 \]

\[ N \frac{\beta}{1 + \beta} = N \frac{1}{(1 + \beta)(1 + R)} \]

\[ \beta(1 + R) = 1 \]

When \( \beta = 1 \), \( R^* = 0 \) and \( c_{A,1}^* = c_{A,2}^* = c_{B,1}^* = c_{B,2}^* = 0.5 \)
The equilibrium with constant and equal consumption levels for all agents $j \in J$ is the result of two assumptions:

- No aggregate volatility (groups of equal size)
- No discounting ($\beta = 1$)

**Useful exercises:**

- Derive the equilibrium allocation when $N_B = 2N_A$
- Derive the equilibrium allocation when $\beta = 0.75$
Extensions

- Borrowing constraints
- Uncertainty
- $N$ periods
Credit constraints

So far, we have assumed the existence of frictionless credit markets. In reality, financial markets are far from perfect and many agents face borrowing constraints.

The simplest way to introduce such constraints is to impose the restriction

\[ b_t \geq 0 \ \forall t \]

More realistic settings would make the borrowing limit a function of wealth (collateral) or lifetime income. But I leave these issues for later.
Example

\[ \text{Max}_{\{c_1, c_2, b_1\}} \quad \ln(c_1) + \beta \ln(c_2) \]
\[ c_1 + b_1 = y_1 \]
\[ c_2 \leq y_2 + (1 + R)b_1 \]
\[ b_1 \geq 0 \]
Example

\[ \mathcal{L} = \ln(c_1) + \beta \ln(c_2) + \lambda (y_1 - c_1 - b_1) + \mu (y_2 + (1 + R)b_1 - c_2) + \gamma b_1 \]

\[ \frac{\delta \mathcal{L}}{\delta c_1} = \frac{1}{c_1} - \lambda = 0 \]
\[ \frac{\delta \mathcal{L}}{\delta c_2} = \frac{\beta}{c_2} - \mu = 0 \]
\[ \frac{\delta \mathcal{L}}{\delta b_1} = -\lambda + \mu(1 + R) + \gamma \leq 0 ; \quad \frac{\delta \mathcal{L}}{b_1} b_1 = 0 \]
\[ \frac{\delta \mathcal{L}}{\delta \gamma} = b_1 ; \quad \gamma \frac{\delta \mathcal{L}}{\delta \gamma} = \gamma b_1 = 0 \]

The third restriction reduces to:

\[ b_1 \left[ \frac{\beta}{c_2} (1 + R) + \left( \gamma - \frac{1}{c_1} \right) \right] = 0 \]
The solution is straightforward

- If the agent’s unconstrained choice is $b_1 > 0$, then $\gamma = 0$ and the optimal allocation is defined by
  
  $$c_2 = \beta(1 + R)c_1$$

- On the contrary, when the agent’s unconstrained choice would be $b_1 \leq 0$, then $c_1 = y_1$ and $c_2 = y_2$. 

Excess sensitivity of consumption

Inspection of consumption data reveals that consumption exhibits a much stronger correlation with current income than what would be suggested by models of frictionless credit markets.

One possible explanation for this “excess sensitivity of consumption” is the existence of borrowing constraints.

Binding credit constraints force agents to lower their consumption in periods with relatively low income, leading to a positive correlation between consumption and current income.
Uncertainty

Let us assume that agents face uncertainty about their second-period income. With probability \( p_l \) the agent receives a low income \( y_{2l} \) and with complementary probability \( p_h = 1 - p_l \) his income is \( y_{2h} > y_{2l} \).

Assuming the agent wishes to maximize expected utility we obtain

\[
\max u(c_1) + \mathbb{E}u(c_{2i}) = c_1 + b_1 = y_1, \quad c_{2,i} = y_{2,i} + (1 + R)b_1.
\]
Uncertainty

The optimization problem of the agent reduces to:

$$\max u(y_1 - b_1) + \beta [p_l u(y_2, l + (1 + R)b_1) + (1 - p_l)u(y_2, h + (1 + R)b_1)]$$

The F.O.C.

$$u'(c_1) = \beta (1 + R) [p_l u'(y_2, l + (1 + r)) + (1 - p_l)u'(y_2, h + (1 + R)b_1)]$$

$$= \beta (1 + R) E u'(c_2)$$

$$< \beta (1 + R) u'(E c_2)$$

where the last inequality follows from Jensen’s inequality.
Risk aversion

Risk aversion is the reluctance of a person to accept a bargain with an uncertain payoff rather than an uncertain bargain with a more certain, but possibly lower, expected payoff.

**Example:** A person is offered a gamble with a payoff of 100 with probability 0.5 and 0 with probability 0.5.

The amount of euros the agent is willing to accept instead of the bet is the *certainty equivalent*. The difference between the *expected payoff* (50) and the certainty equivalent is the *risk premium*.
The higher the curvature of the utility function, the higher the risk aversion. Two popular measures of risk aversion are:

- **Arrow-Pratt measure of Absolute Risk-Aversion**
  \[ A(c) = - \frac{u''(c)}{u'(c)} \]

- **Coefficient of Relative Risk Aversion**
  \[ R(c) = - \frac{cu''(c)}{u'(c)} \]

When preferences are DARA, a richer agent will invest a larger amount in risky assets than a poor agent. Also, CRRA utility implies DARA.
N periods

Let us now assume that the economy lasts for \( N > 2 \) periods. For simplicity, we return to a deterministic setting and we denote the terminal period by \( T \).

\[
\max_{\{c_t, b_t\}} \sum_{t=0}^{T} \beta^t u(c_t)
\]

subject to:

\[
c_t + b_t = y_t + (1 + R_t)b_{t-1}
\]

Like before we will impose a condition to avoid that agents spend more than their lifetime income.
N periods

Let $\lambda_t$ denote the Lagrange multiplier associated with the period-$t$ budget constraint.

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left[ u(c_t) + \lambda_t[y_t + (1 + R_t)b_{t-1} - c_t - b_t] \right]$$

F.O.C.

$$u'(c_t) = \lambda_t$$

$$\beta^t \lambda_t = \beta^{t+1} \lambda_{t+1}(1 + R_t)$$

Take any two consecutive periods $t$ and $t + 1$ and solve:

$$u'(c_t) = \beta(1 + R_t)u'(c_{t+1})$$
Generalizing lifetime budget constraints

Consider the following sequence of budget constraints

\[ c_0 + b_0 = y_0 \]
\[ c_1 + b_1 = y_1 + (1 + R)b_0 \]
\[ \vdots \]
\[ c_T + b_T = y_T + (1 + R)b_{T-1} \]

Solving forwards, we obtain:

\[ b_T = \sum_{t=0}^{T} \frac{y_t}{(1 + R)^t} - \sum_{t=0}^{T} \frac{c_t}{(1 + R)^t} \]

When life ends in \( T \), \( b_T = 0 \) and we obtain the standard expression.
Alternative specification

Since the agent can “freely” move consumption between periods, we can write the optimization problem as:

\[ L = \sum_{t=0}^{T} \beta^t u(c_t) + \lambda \left( \sum_{t=0}^{T} \left[ \frac{y_t}{(1+R)^t} - \frac{c_t}{(1+R_t)^t} \right] \right) \]

For any representative period \( t \) we obtain the following F.O.C.:

\[ \beta^t u'(c_t) = \lambda \frac{1}{(1+R_t)^t} \]

Taking any two consecutive periods \( t \) and \( t + 1 \), we arrive once more at the Euler equation.
Aggregation

In our stylized model with identical agents it is trivial to derive the aggregate demand function.

With heterogeneous agents, aggregation is far from trivial, but we leave this for later.
Permanent income hypothesis (PIH)

Consider an infinitely-lived agent with the following present value of lifetime income:

\[ W_t = Y_t + \frac{Y_{t+1}}{1+R} + \frac{Y_{t+1}}{(1+R)^2} + \frac{Y_{t+2}}{(1+R)^2} + \ldots + \frac{Y_{t+s}}{(1+R)^{t+s}} + \ldots \]

Milton Friedman defines the permanent income of the agent as the maximum level of income the agent can afford in each period of his lifetime without altering the value of his wealth \( W_t \).

To be more precise, the agent can save \( W_t/(1+R) \) and consume his permanent income

\[ Y_p = \frac{R}{1+R} W_t \]

Notice, however, that \( Y_p \) need not correspond to the optimal consumption choice of the agents! In general \( C_p = g(Y_p, R) \).
Permanent Income Hypothesis

Assume that in each period, income and consumption have a transitory component

\[ Y_t = Y_p + Y_{tr,t} \]
\[ C_t = C_p + C_{tr,t} \]
\[ C_p = k(\ldots) Y_p \]

with \( \mathbb{E} Y_{tr} = 0 \) and \( \text{Cov}(Y_p, Y_{tr}) = 0 \). In that case, \( \text{Cov}(C_{tr,t}, Y_{tr,t}) = 0 \).
Random Walk Hypothesis

Bob Hall reconsidered the results of the permanent income hypothesis (PIH) assuming that agents have rational expectations — agents make efficient use of all the available information.

**Main prediction:** If the PIH holds and agents have rational expectations then consumption changes should be unpredictable. That is, consumption follows a random walk and $E_t c_{t+1} = c_t$.

**In ordinary words:** Agents only revise their consumption decisions if they receive new information that forces them to revise their expectations about $Y_p$. This is intrinsically unpredictable.
Life Cycle Hypothesis

- Nobel prize winner Franco Modigliani considered an extension of the basic model of Fisher to analyze decisions along the life cycle.
- The basic insight: income varies in an almost deterministic manner along the life cycle and agents use the credit market to insulate consumption from these movements in income.
- Agents typically borrow when they are young (education, housing), save during prime age and dissave during retirement.
- At different stages of their life, agents or households therefore act at different sides of the credit market.
The Life Cycle Hypothesis

Ingreso y consumo

Perfil de ingresos a lo largo de la vida

Edad
The Life Cycle Hypothesis

Once again, when the agents have unrestricted access to credit markets, consumption decisions no longer depend on income in a given period but on $Y_p$.

- Savings tend to be negative in periods in which $Y_t < Y_p$.
- Savings tend to be positive in periods in which $Y_t > Y_p$.

These optimal savings decisions allow agents to maintain a constant (or smooth) consumption level although income changes.