The Labour-Leisure Choice

Máster en Economía Internacional

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Motivation

- So far, we have analyzed consumption-savings decisions in a pure exchange economy.
- This Lecture introduces endogenous labour and production.
- We will consider two environments:
  - Autarky — the agent operates a backyard technology and consumes his own output.
  - Competitive equilibrium — Agents offer their labour services on a perfectly competitive labour market.
- We will show that both environments yield identical outcomes.
Setup

- For the moment we assume a static environment. Alternatively, one could think of an economy with perishable goods.

- The agents have 1 unit of time and need to decide how much to work, $l$, and how much leisure time, $1 - l$, they wish to enjoy.

- The instantaneous utility of agents is defined over consumption and leisure and is additively separable:

$$U(c, 1 - l) = u(c) + v(1 - l)$$

with $u'(\cdot) > 0$, $u''(\cdot) < 0$, $v'(\cdot) > 0$, $v''(\cdot) < 0$.

- In an extension we will also consider the possibility in which utility depends on consumption and effort, $l$. 

Two environments

Throughout this theme we will compare the outcomes in two environments:

- **Autarky** — the agent consumes his own output

  \[ c = f(A, l) \]

- The agent offers his labour services on a competitive labour market in return of a wage, \( w \), and may have income from capital

  \[
  c = \begin{cases} 
  w l & \text{No capital} \\
  w l + r & \text{Each agent has one unit of capital}
  \end{cases}
  \]
The Marginal Rate of Substitution of Consumption for Leisure

Suppose we raise $l$ by an infinitesimally small amount $(dl)$. The MRS tells us by how much we need to raise $c$ to keep the level of utility constant. Taking the total derivative,

$$d\bar{u} = 0 = u'(c)\, dc + v'(1 - l)\, (-1)\, dl$$

$$u'(c)\, dc = v'(1 - l)\, dl$$

utility gain

utility loss

$$MRS = \frac{dc}{dl} = \frac{v'(1 - l)}{u'(c, 1 - l)} > 0$$
Indifference curves

Curvas de indiferencia

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Labor-Leisure decisions

Autarky

In autarky the agent solves the following problem:

$$\begin{align*}
\max_{c,l} & \quad u(c, 1 - l) \\
\text{s.t.} & \quad c = f(l)
\end{align*}$$

The best option is to solve this problem with the help of the substitution method. Replacing $c$ in the maximand by $f(l)$, we obtain:

$$\max_{l} \left[ u(f(l), 1 - l) \right]$$
The optimal choice

- The first-order condition (FOC) that characterizes the optimal choice is given by:

\[ u_1(f(l), 1 - l) f'(l) + u_2(f(l), 1 - l) (-1) = 0 \]

\[ \underbrace{u_1(f(l), 1 - l) f'(l)}_{\text{marginal benefit}} = \underbrace{u_2(f(l), 1 - l)}_{\text{marginal cost}} \]

\[ MRS = \frac{u_2(c, 1 - l)}{u_1(c, 1 - l)} = f'(l) = MPL \]

Notice that MPL can be interpreted as the marginal rate of transformation of labor in goods.
Gráfica de representación

La gráfica muestra la elección entre trabajo y consumo. La curva de consumo (c) y la curva de trabajo (l) se representan en un espacio bidimensional. La curva de consumo se evalúa a través del punto B, que representa el equilibrio entre trabajo y consumo. La función de consumo se denota como c = y = f(l).

La curva de utilidad (u1) y la curva de utilidad (u2) se cruzan en el punto B, lo que indica la optimización de la elección entre trabajo y consumo.
The Lagrange method

The alternative is to use the Lagrange method. The Lagrangian associated with our optimization problem can be written as

\[ L = u(c) + v(1 - l) + \lambda [f(l) - c] \]

where \( \lambda \) denotes the Lagrange multiplier. The FOCs are given by:

\[
\frac{\partial L}{\partial c_t} = u'(c) - \lambda = 0
\]
\[
\frac{\partial L}{\partial l_t} = v'(1 - l)(-1) + \lambda f'(l_t) = 0
\]
\[
\frac{\partial L}{\partial \lambda} = f(l_t) - c_t = 0
\]

As \( \lambda = u'(c) \), we again obtain the standard optimality condition

\[
\frac{v'(1 - l)}{u'(c)} = f'(l)
\]
Examples

Capital in the production function (Cobb-Douglas with $\kappa = 1$)

$$\max_{c,l} \left[ \ln(c) + \ln(1-l) \right]$$

subject to

$$c = Al^\alpha$$

No capital

$$\max_{c,l} \left[ \ln(c) + \ln(1-l) \right]$$

subject to

$$c = Al$$
Our next objective is to study the effects of productivity shocks on the optimal labour supply decisions.

In general, productivity shocks produce both income and substitution effects.

In modern real business cycle models (RBC) this type of shocks is the driver of business cycle movements.
Example I: Parallel shift of the production function
Example II: A positive TFP shock
Definitions

**Income effect:** An increase in output for any given choice of $l$, but no change in the MPL schedule.

**Substitution effect:** A change in the MPL, or equivalently the marginal rate of transformation of labor into consumption.
Example III: A Pure Income Effect

\[ c = y = f(l) \]

- Graph showing the Labour-Leisure Choice with consumption (c) on the y-axis and work (l) on the x-axis.
- Points A and B represent different consumption levels with corresponding work hours.
- The graph illustrates the trade-off between consumption and work hours.
Example IV: The Effects of a Pure Substitution Effect
Combined Income and Substitution Effects

Suppose that Robinson has a fixed endowment \( a \geq 0 \) of consumption goods and that \( f(I) = AI \). Accordingly,

\[
c = a + AI
\]

and

\[
\max_I \left[ \log (a + AI) + \log (1 - I) \right]
\]

Notice that the FOC for this problem can be written as

\[
\frac{1}{a + AI} A = \frac{1}{1 - I}
\]

And so,

\[
I^* = \frac{A - a}{2A}
\]

\[
c^* = \frac{A}{2} + \frac{a}{2}
\]
Combined Income and Substitution Effects

In our example,

- **The increase in** $a$ generates a pure income effect. In response to this positive income effect, Robinson increases both leisure and consumption.

- **The increase in** $A$ produces both income and substitution effects. The latter dominate as Robinson’s optimal labor supply is rising in $A$. 
Disentangling Income and Substitution Effects

\[ y = f(l) \]

\[ \text{Consumo} \]

\[ c \]

\[ c_0 \]

\[ c_1 \]

\[ c_2 \]

\[ \text{Trabajo} l \]

\[ l_0 \]

\[ l_1 \]
Competitive labour market

Let us now consider the alternative case in which agents offer their labour services on a competitive labour market:

\[
\begin{align*}
\max_{c,l} & \quad [u(c) + v(1 - l)] \\
\text{s.t} & \quad c = w l
\end{align*}
\]

It is straightforward to show that the optimal solution satisfies

\[
\frac{v'(1 - l)}{u'(c)} = w
\]

This yields the same solution as in autarky as long as \( w = PML \).
Example: logarithmic utility

\[
\max_{c,l} \left[ \ln(c) + \rho \ln(1 - l) \right]
\]

s.t
\[c = wl\]

F.O.C.

\[c : \quad \frac{1}{c} = \lambda\]
\[l : \quad \frac{\rho}{1 - l} = \lambda w\]
\[\lambda : \quad c = wl\]

Solutions: \(l^* = \frac{1}{1+\rho}; c^* = \frac{A}{1+\rho}\)
Market Equilibrium

- For simplicity, we consider the case of a linear production technology (no capital)
  \[ Y = AL^f \]

- In this economy the representative firm solves the following problem
  \[ \Pi = AL^f - wL^f \]

- \[ \frac{\delta \Pi}{\delta L} = A - w \leq 0, \quad \frac{\delta \Pi}{\delta L} L = 0 \]

- Equilibrium: \( w = A \)
The household’s problem

The individual households solve the following problem

$$\max_{c,l} \left[ \ln(c) + \rho \ln(1 - l) \right]$$

subject to

$$c = wl$$

Yet, in equilibrium we know that $w = A$ and so the agent solves the same problem as in autarky.

$$l^* = \frac{1}{1 + \rho} \ ; \ c^* = \frac{A}{1 + \rho}$$
Equilibrium

In this setting, the proof of existence is trivial. Suppose there are $N$ identical agents in this economy

- Labour supply is perfectly elastic: $L^s = NI^* = N \frac{1}{1+\rho}$
- Labour demand $L^f$ is perfectly elastic at $w = A$.

$$Y^* = A(NI^*) = \frac{AN}{1+\rho} = C^*$$
The tax wedge

A proportional tax on labour income introduces a wedge between the marginal rate of substitution of consumption for leisure and MPL.

\[
\max_{c,l} \left[ u(c) + \nu(1 - l) \right] \\
\text{s.t} \\
\quad c = w(1 - \tau)l
\]

In this setting, the optimal choice is characterized by

\[
\frac{\nu'(1 - l)}{u'(c)} = (1 - \tau)w = (1 - \tau)MPL
\]

where the last equality is satisfied if the labour market is perfectly competitive.
Assuming the agents have access to a perfectly competitive credit market, we can write the optimization problem as

\[ L = \sum_{t=0}^{N} \beta^t [u(c_t) + v(1 - l_t) + \lambda_t [w_t l_t + (1 + R) b_{t-1} - c_t - b_t]] \]

\[ u'(c_t) = \lambda_t \]
\[ v'(1 - l_t) = \lambda_t w_t \]
\[ \beta^t \lambda_t = \beta^{t+1} \lambda_{t+1} (1 + R) \]

Combining the first two F.O.C.s we obtain the standard condition

\[ \frac{v'(1 - l_t)}{u'(c_t)} = w_t \]
Taking the ratio between the FOC for $c_{t+1}$ and $c_t$ we obtain,

$$\frac{u'(c_{t+1})}{u'(c_t)} = \frac{\lambda_{t+1}}{\lambda_t} = \frac{1}{\beta(1 + R)}$$

where we have used the third FOC to solve for $\lambda_{t+1}/\lambda_t$.

This leads to the well-known Euler consumption equation:

$$u'(c_t) = \beta(1 + R)u'(c_{t+1})$$
Similarly, taking the ratio between the FOCs for $l_{t+1}$ and $l_t$, we obtain

$$\frac{v'(1 - l_{t+1})}{v'(1 - l_t)} = \left( \frac{\lambda_{t+1}}{\lambda_t} \right) \left( \frac{w_{t+1}}{w_t} \right)$$

which can be rewritten as:

$$\beta(1 + R) \frac{v'(1 - l_{t+1})}{v'(1 - l_t)} = \left( \frac{w_{t+1}}{w_t} \right)$$

The above equation implicitly defines the ratio $l_{t+1}/l_t$ as an increasing function of the wage ratio. If $\beta(1 + R) = 1$ the agents want to perfectly smooth consumption.

If in addition $w_{t+1}/w_t = 1$, they also want to work the same hours in two consecutive periods.
\[ \beta(1 + R) \frac{v'(1 - l_{t+1})}{v'(1 - l_t)} = \left( \frac{w_{t+1}}{w_t} \right) \]

The above solution for the optimal labour supply decisions also suggests that \( l_{t+1}/l_t \) is a function of the interest rate.

More precisely, an increase in \( R \) generates a fall in \( l_{t+1}/l_t \).

A rise in the interest rate reduces the present value of future wages, \( w_{t+1}/(1 + R) \) and makes work in period \( t \) more attractive.
Labour supply in two periods

\[ L = \sum_{t=1}^{2} \left( \beta^{t-1} \left[ \log(c_t) - \phi \frac{l_{t}^{1+\theta}}{1+\theta} \right] + \lambda \left[ w_1 l_1 + \frac{w_2 l_2}{1+R} \right] \right) \]

\[ \frac{1}{c_1} = \lambda \]

\[ \frac{\beta}{c_2} = \frac{\lambda}{1 + R} \]

\[ \phi l_1^{\theta} = \lambda w_1 = \frac{w_1}{c_1} \]

\[ \phi l_2^{\theta} = \frac{\lambda w_2}{\beta (1 + R)} = \frac{w_2}{c_2} \]
Solutions

Combining the FOCs for the consumption levels we find

\[ c_1 = \frac{1}{1 + \beta} \left[ w_1 l_1 + \frac{w_2 l_2}{1 + R} \right] \]

Similarly, from the FOCs for labor supply, it follows that

\[ \left( \frac{l_2}{l_1} \right)^\theta = \frac{w_2}{\beta(1 + R) w_1} \]

And so,

\[ l_2 = \left[ \frac{w_2}{\beta(1 + R) w_1} \right]^{\frac{1}{\theta}} l_1 \]
Solutions

\[ \phi l_1^\theta = \frac{w_1}{c_1} \]

\[ \phi l_1^\theta \left[ w_1 l_1 + \frac{w_2 l_2}{1 + R} \right] = w_1 (1 + \beta) \]

\[ \phi l_1^\theta \left[ l_1 + \frac{w_2}{w_1 (1 + R)} \left( \frac{w_2}{\beta (1 + R) w_1} \right)^{\frac{1}{\theta}} l_1 \right] = 1 + \beta \]

\[ \phi l_1^{1+\theta} \left[ 1 + \left( \frac{w_2}{(1 + R) w_1} \right)^{1+\frac{1}{\theta}} \beta^{-\frac{1}{\theta}} \right] = 1 + \beta \]
Two important elasticities

There are two important elasticities we should care about:

- **Intertemporal elasticity of substitution (IES) for labor supply**: The elasticity of relative labour supply across periods with respect to the present value of wage growth.

- **Frisch elasticity**: The elasticity of labour supply with respect to the wage for a constant marginal utility of wealth (measures pure substitution effect).
Recall that the intertemporal optimality condition for labour supply can be written as:

$$\frac{\beta v' (1 - l_{t+1})}{v'(1 - l_t)} = \frac{w_{t+1}}{(1 + R) w_t} = \tilde{W}_t$$

The EIS measures the elasticity of $(l_{t+1}/l_t)$ with respect to $\tilde{W}_0$.

In our last example

$$\left( \frac{l_{t+1}}{l_t} \right)^\theta = \frac{1}{\beta} W_t$$

In this case, the EIS is $1/\theta$. 

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The Frisch elasticity can be written as:

\[
\frac{v'(l_t)}{l_t v''(l_t)}
\]

In our example, \(v'(l_t) = \phi l^\theta\) and \(v''(l_t) = \phi \theta l^{\theta-1}\) and so the Frisch elasticity is

\[
\frac{\phi l^\theta}{\phi \theta l^{\theta-1}l_t} = \frac{1}{\theta}
\]