1. Consider a continuous-time version of the Solow growth model with population growth and no technological progress. Households own the capital stock and save a fraction \( \sigma \in (0, 1) \) of their income. The production technology of the representative firm is described by

\[
Y(t) = AK(t)^\alpha L(t)^{1-\alpha},
\]

where \( A \) denotes total factor productivity, \( K(t) \) denotes the capital stock at time \( t \), and \( L(t) \) denotes both the size of the population and the number of workers. We assume that \( \dot{L}(t)/L(t) = n \), while capital is assumed to depreciate at rate \( \delta \). The factor markets are competitive and we denote the competitive wage and the rental price of capital at time \( t \) by \( w(t) \) and \( R(t) \), respectively.

a. Write down the expression for \( \dot{K}(t) \) and use this expression to derive the expression for \( \dot{\kappa}(t) \) where \( \kappa(t) = K(t)/L(t) \) is the capital-labour ratio at \( t \).

b. Define the full equilibrium path of this economy. Let \( K(0) \) denote the initial value of the capital stock.

c. Derive the expression for the steady-state value of the capital-labour ratio, \( \kappa^* \) and calculate the value \( \sigma \) that leads to a steady state that coincides with the steady-state allocation of the Golden Rule. Denote this value of the savings rate by \( \sigma^{GR} \).

d. Steady-state allocations with a capital-labour ratio \( \kappa^* \) that is larger than the Golden Rule capital-labour ratio \( \kappa^{GR} \) are characterized as dynamically inefficient allocations. Explain why and demonstrate that in these economies the net-return on capital, \( r(t) = R(t) - \delta \) is smaller than \( n \).

e. Analyze the implications of an increase in the population growth rate for the steady-state values of \( \kappa \) and \( c \) when \( \sigma = \sigma^{GR} \). Illustrate your answer with a figure.
2. Consider a pure exchange economy that lasts for two periods. There are two types of agents indexed by $i = a, b$ who receive a known endowment $z_{i,t}$ in both periods. The utility function of both agents is given by

$$U(c_{i,1}, c_{i,2}) = \frac{c_{i,1}^{1-\sigma} - 1}{1-\sigma} + \beta \frac{c_{i,2}^{1-\sigma} - 1}{1-\sigma}$$

All the agents have access to a perfectly competitive credit market with interest rate $r$.

a. Solve the agents’ optimization problem for arbitrary values of the parameters and the equilibrium interest rate.

b. Derive the equilibrium interest rate when both groups are of equal size, $\beta = 1$ and $z_{a,1} = z_{b,2}$ and $z_{a,2} = z_{b,1} = 0$.

c. Now suppose that there are $n > 1$ agents of type $b$ per agent of type $a$. Derive the expression for the equilibrium interest rate and explain how it depends on the value of the elasticity of intertemporal substitution.

d. Return to the example in b but now suppose that all the agents in the economy are subject to the borrowing constraint $b_1 > 0$. Derive the equilibrium allocation and explain how the borrowing constraint affects the welfare of both groups of agents.

e. Now assume that the economy lasts for many periods and that the income stream of the agents suffers (deterministic) changes over time. How can the agents avoid having to reduce consumption in periods with low income?
2. Consider a pure exchange economy of two periods. All agents have the following preferences over current and future consumption, $c_1$ and $c_2$ respectively:

$$U(c_1, c_2) = \sqrt{c_1} + \beta \sqrt{c_2},$$

with $0 < \beta \leq 1$. The $N_a$ agents of type $a$ receive a known endowment of 1 unit of the final good in period 1 and 0 in period 2, i.e. $y^a_1 = 1$ and $y^a_2 = 0$. Similarly, the $N_b$ agents of type $b$ receive an endowment stream of $y^b_1 = 0$ and $y^b_2 = 1$. The agents have access to a perfectly competitive credit market and we denote the real interest rate by $R$.

a. Derive the expression for the Inter-temporal Marginal Rate of Substitution $dc_2/dc_1$ (10 points).

b. Solve the optimization problem of a representative agent for arbitrary values of $R$, $\beta$ and the present value of lifetime resources, $x$. Your answer should include the expressions for the optimal consumption levels $c^*_1$, $c^*_2$ and bond holdings $b^*_1$ (or savings) (15 points).

c. Write down the equilibrium condition for the credit market and derive the equilibrium interest rate $R$ for the case in which $N_b = N_a$ and $\beta = 1$. How much do the agents consume in both periods? (15 points)

d. Identify the necessary condition for an equilibrium in which $\beta(1 + R) = 1$ and all agents choose a constant consumption profile (10 points).