A Canonical OLG model with Capital

Máster en Economía

Universidad Autonóma de Madrid

November 2014
OLG with physical capital

- So far, we have considered two-period economies.

- These models delivered important insights. But for many purposes it is important to consider economies with an infinite horizon. In particular, with infinite horizons we can derive well-defined steady states (there is no terminal point).

- In modern macroeconomics, steady state allocations are normally the benchmark for policy analysis.

- At the end of this course we will analyze models with infinitely-lived agents.

- On the contrary, in this Lecture we consider the alternative of an OLG model in which the agents live during two periods.
A canonical OLG model

- OLG models can be constructed for many purposes.

- In this Lecture we will consider a canonical version of the OLG model with capital that delivers a similar equilibrium path as the Solow model.

- The difference: savings and consumption decisions are endogenous.

- Hence the model can be used to perform welfare analysis and we can study the optimal response of agents to policy changes.

- Besides that we’ll see that the model can be used to illustrate the main features of a rudimentary RBC model.
The basic setup

- We will consider an economy in which agents (households) live for two periods.
- The agents of generation $t$ are young in $t$ and old in $t + 1$.
  - When young, they coincide with the old agents of generation $t - 1$.
  - When old, they live with the young generation of $t + 1$.
- There is a unique good that can either be consumed or saved as capital.
- For the moment each dying household is replaced by one young household. At any moment in time there are $N$ young households, and $N$ old households.
Labor supply and savings

For the moment we assume that labor supply is inelastic. Moreover, the agents only work during the first period of their lives.

- Each young agent offers one unit of labor to a representative firm.
  - Labor supply is completely inelastic. The young agents only need to make a decision about their savings and consumption.
  - Aggregate savings of the young serve as the stock of capital in the next period.

- The old are retired and do not work.
  - They rent their savings (capital) to the representative firm and consume all their resources before they die.
In each period there is a representative firm that hires capital from the old and labor services from the young to produce the unique final good. The firm is endowed with a neo-classical production function:

\[ Y_t = A_t F(K_t, L_t). \]

The firm pays a competitive wage to the young and a rental price (interest) to the owners of capital. Thus, profits are equal to

\[ \Pi = A_t F(K_t, L_t) - w_t L_t - r_t K_t. \]
Before we proceed, it is important to highlight the assumptions of physical capital.

- Capital is owned by households and rented to firms in a competitive market. This assumption is standard in dynamic macromodels.

- Capital depreciates at rate $\delta$ — in each period a percentage $\delta$ of the capital stock “melts” away.

- Gross investment is equal to the savings of the young. Net-investment is gross investment minus depreciation.
Consumption and savings decisions
The intertemporal budget constraint

- Let's consider the problem of a representative agent of generation $t$. The agent offers one unit of labor and receives a wage of $w_t$. Hence, the income of the young is simply $w_t$.

- The agent needs to decide how much he wants to consume this period, $c_{yt}$, and how much he wants to save for his old age, $s_{t+1}$. Note that the savings of generation $t$ carry a subscript $t + 1$. These are goods saved for the future.

- Hence, the first-period budget constraint can be written as:

\[
 w_t = c_{yt} + s_{t+1} \\
 s_{t+1} = w_t - c_{yt}
\]
The inter-temporal budget constraint

- As mentioned before, the old agents of any generation consume all their resources:

\[
c_{ot+1} = (1 - \delta)s_{t+1} + s_{t+1}r_{t+1} = s_{t+1}(1 + r_{t+1} - \delta)
\]

- The old receive the amount of capital rented to firms \((s_{t+1})\) plus interest \((s_{t+1}r_{t+1})\). Moreover, capital depreciates at rate \(\delta \in (0, 1)\) per period. Hence \((1 - \delta)s_{t+1}\) is the value of capital net of depreciation.

- The old consume the net value of their savings \((1 - \delta)s_{t+1}\) plus interest \((s_{t+1}r_{t+1})\). Total consumption is \(s_{t+1}(1 + r_{t+1} - \delta)\)
The inter-temporal budget constraint

- Combining the budget constraints for the two periods we get

\[ c_{ot+1} = s_{t+1} (1 + r_{t+1} - \delta) \]
\[ = (w_t - c_{yt}) (1 + r_{t+1} - \delta) \]

- Hence, the inter-temporal budget constraint for a member of generation \( t \) is given by

\[ \frac{c_{ot+1}}{(1 + r_{t+1} - \delta)} = (w_t - c_{yt}) \]
\[ c_{yt} + \frac{c_{ot+1}}{(1 + r_{t+1} - \delta)} = w_t \]

- In words, the discounted present value of lifetime consumption is equal to the wage.
The inter-temporal budget constraint

\[ c_{ot+1} \]

\[ w_t (1 + r_{t+1} - d) \]

\[ (w_t - c_{yt})(1 + r_{t+1} - d) \]

**Figure 2a**
Preferences

- As usual, we represent the preferences of the agent by means of a utility function:

\[ U(c_{yt}, c_{ot+1}) = u(c_{yt}) + \beta u(c_{ot+1}), \text{ with } \beta < 1, \ u' > 0, \ u'' < 0, \]

- In most cases we will consider the example of logarithmic preferences

\[ U(c_{yt}, c_{ot+1}) = \log(c_{yt}) + \beta \log(c_{ot+1}) \]
There is a large number of firms that combine capital and labor to produce the unique final good using a technology that is represented by

\[ Y_t = A_t F(K_t, L_t), \]

The neo-classical production function \( F \) is strictly increasing and concave in both arguments:

\[
\frac{\partial F_t}{\partial K_t} > 0, \quad \frac{\partial F_t}{\partial L_t} > 0, \quad \frac{\partial^2 F_t}{\partial K_t^2} < 0, \quad \frac{\partial^2 F_t}{\partial L_t^2} < 0.
\]

\( F \) also exhibits constant returns to scale (CRS), i.e.

\[ F(\lambda K_t, \lambda L_t) = \lambda F(K_t, L_t), \quad \text{for all } \lambda > 0, \]

and satisfies the Inada conditions

\[
\lim_{K \to 0} F_K = \lim_{L \to 0} F_L = \infty, \quad \text{and} \quad \lim_{K \to \infty} F_K = \lim_{L \to \infty} F_L = 0.
\]
The representative firm

- Since the production function exhibits CRS, we can assume **without loss of generality** that there is a single representative firm. The objective of the firm is to maximize its profits

\[
\max_{L_t, K_t} Y(K_t, L_t) - w_t L_t - r_t K_t.
\]

- The FOC’s are given by

\[
\begin{align*}
w_t &= \frac{\partial Y_t}{\partial L_t} = MPL; \\
r_t &= \frac{\partial Y_t}{\partial K_t} = MPK
\end{align*}
\]

- The representative firm hires capital (labor) until the marginal product of capital (labor) is equal to the rental price of capital (wage).
Profits

- The assumption of a CRS production function guarantee that the equilibrium profits of the representative firm are equal to zero.

- This result is a direct consequence of the so-called Euler theorem which establishes that if \( F(K_t, L_t) \) is homogenous of degree 1, then

\[
F(K_t, L_t) = \frac{\partial F}{\partial K_t} K_t + \frac{\partial F}{\partial L_t} L_t
\]

\[
= r_t K_t + w_t L_t
\]

- For example, when \( F(K, L) = AK^\alpha L^{1-\alpha} \) we have

\[
F_K K + F_L L = [\alpha AK^{\alpha-1} L^{1-\alpha}]K + [(1 - \alpha)AK^\alpha L^{-\alpha}]L = F(K, L)
\]
Timeline of decisions

<table>
<thead>
<tr>
<th>Período t-1</th>
<th>Período t</th>
<th>Período t+1</th>
</tr>
</thead>
<tbody>
<tr>
<td>[jóvenes] generación t-1</td>
<td>[jóvenes] generación t</td>
<td>[viejos]</td>
</tr>
<tr>
<td>Reciben salario ( (w_{t-1}) ), Consumen ( (c_{t-1}) ), Ahorran ( (s_t) )</td>
<td>Reciben salario ( (w_t) ), Consumen ( (c_t) ), Ahorran ( (s_{t+1}) )</td>
<td>Proveen capital a la firma ( (s_t) )</td>
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FIRMA

\[ \text{capital} \uparrow (1+r_{t-d}) s_t \downarrow \text{consumo} \]

Proveen capital a la firma \( (s_t) \)
Consumen el ingreso total \((1+r_{t-d}) s_t\)
The household problem

- Agents born in $t$ ($\forall t \geq 1$) solve the following problem:

$$\max \; u(c_y, t) + \beta \; u(c_o, t+1)$$

s.t.  
$$c_y, t + s_{t+1} \leq w_t,$$

$$c_{o, t+1} \leq (1 + r_{t+1} - \delta) \; s_{t+1}.$$  

- Combining the two budget constraints, we obtain the familiar problem:

$$\max_{c_y, t, c_o, t+1} \; u(c_y, t) + \beta \; u(c_o, t+1)$$

s.t.  
$$c_y \; t + \frac{c_o \; t+1}{1+r_{t+1}-\delta} = w_t.$$
The household problem

- The Langrangean associated with the household problem is given by

\[
L = \max_{c_{yt}, c_{ot+1}} u(c_{yt}) + \beta u(c_{ot+1}) + \lambda \left[ w_t - c_{yt} - \frac{c_{ot+1}}{1+r_{t+1}-\delta} \right].
\]

- The FOCs are given by

\[
u'(c_{yt}) = \lambda
\]

\[
\beta u'(c_{ot+1}) = \frac{\lambda}{(1+r_{t+1}-\delta)}.
\]

- The only difference with before is the discount factor which includes the net-return on capital.
**Consumption Euler condition**

In our OLG setup, the consumption Euler condition of generation-\(t\) households is given by:

\[
\beta (1 + r_{t+1} - \delta) u'(c_{o,t+1}) = u'(c_{y,t})
\]

In principle, this equation should include an expectation sign as households need to decide the value of \(s_{t+1}\) before they know the realization of \(r_{t+1}\).

- In a deterministic setting one could assume perfect foresight.
- In a stochastic setting, the common assumption is rational expectations.
- Here we avoid these complications — we only consider examples in which \(s_{t+1}\) does not depend on \(r_{t+1}\).
Our classroom example

- The economy is populated by $N$ individuals of each generation. Hence, in each period there is a population of size $2N$ ($N$ young and $N$ old).

- The agents have logarithmic preferences

$$u(c_{yt}, c_{ot+1}) = \log(c_{yt}) + \beta \log(c_{ot+1}).$$

- The production technology of the representative firm can be represented by a Cobb-Douglas production function:

$$Y_t = A_t K_t^\alpha L_t^{1-\alpha}.$$
The firm problem

Let’s start with the solution of the static firm problem:

$$\max_{K_t, L_t} A_t K_t^\alpha L_t^{1-\alpha} - w_t L_t - r_t k_t,$$

In previous lectures we have seen that the FOC’s are given by:

$$w_t = (1 - \alpha) A_t K_t^\alpha L_t^{-\alpha} = (1 - \alpha) A_t \left( \frac{K_t}{L_t} \right)^\alpha,$$

$$r_t = \alpha A_t K_t^{\alpha - 1} L_t^{1-\alpha} = \alpha A_t \left( \frac{K_t}{L_t} \right)^{\alpha - 1}.$$
The household problem

- Recall that the consumption-savings decision of any generation \( t \geq 1 \) is given by:

\[
\max_{c_{yt}, c_{ot+1}} \log(c_{yt}) + \beta \log(c_{ot+1})
\]

subject to:

\[
c_{yt} + \frac{c_{ot+1}}{1 + \delta} = w_t.
\]

- With logarithmic preferences the consumption Euler condition

\[
\begin{align*}
\left\{ \begin{array}{c}
u'(c_{yt}) = \beta(1 + r_{t+1} - \delta)u'(c_{ot+1}). \\
MC & \text{MC} \\
MB & \text{MB}
\end{array} \right.
\end{align*}
\]

simplifies to

\[
\frac{1}{c_{yt}} = \beta(1 + r_{t+1} - \delta) \frac{1}{c_{ot+1}}.
\]

- Hence,

\[
c_{0t+1} = \beta c_{yt}(1 + r_{t+1} - \delta).
\]
The optimal savings ratio

- To show that the equilibrium savings ratio does not depend on the interest rate, we write the Euler condition as

\[ c_{ot+1} = \beta cyt (1 + r_{t+1} - \delta). \]

- Substituting the above condition in the budget constraint we get:

\[ cyt + \frac{\beta cyt (1 + r_{t+1} - \delta)}{1 + r_{t+1} - \delta} = cyt + \beta cyt = w_t. \]

- Hence,

\[ cyt = \frac{1}{1 + \beta} w_t, \]

\[ s_{t+1} = w_t - \frac{1}{1 + \beta} w_t = \frac{\beta}{1 + \beta} w_t. \]
The household problem

The solution

1. 
\[ c_{yt} = \frac{1}{1 + \beta} w_t, \]  
\hspace{1cm} (3)

2. 
\[ c_{ot+1} = \beta c_{yt} (1 + r_{t+1} - \delta) = \frac{\beta(1 + r_{t+1} - \delta)}{1 + \beta} w_t, \]  
\hspace{1cm} (4)

3. 
\[ s_{t+1} = w_t - c_{yt} = w_t - \frac{1}{1 + \beta} w_t = \frac{\beta}{1 + \beta} w_t. \]  
\hspace{1cm} (5)

NB: savings depend on the value of \( w_t \) and NOT on future values of the interest rate!
Dynamics

- Modern macro-models are characterized by very complicated dynamics as the links between periods run in both directions.
  - Current decisions about savings and investment affect future opportunities and prices.
  - Beliefs about future prices and returns affect current decisions.
- Here we simplify matters by adopting a specification in which equilibrium savings and investment only depend on current prices.
  - Thus the dynamic interactions run in one direction: today’s decisions define the state of the economy tomorrow.
- Nonetheless, the dynamics are interesting as they mimic the dynamics of the Solow growth model.
  - The savings ratio of the young is constant and after an initial period of growth the economy converges to a steady state with zero growth.
  - Short-term (fluctuations) and long-term phenomena (growth) can be analyzed in the same model.
The equilibrium of the economy

The resource constraint

- Note that the total resources in the economy in period $t$ are equal to

$$A_t K_t^\alpha L_t^{1-\alpha} + (1 - \delta) K_t$$

- In equilibrium these resources are used for the consumption of the old in period $t$, $N_{c_{ot}}$, consumption of the young in the same period $N_{c_{yt}}$, and savings of the young, $N_{s_{t+1}}$.

- Accordingly, the equilibrium condition for the goods market can be written as:

$$N_{c_{yt}} + N_{c_{ot}} + N_{s_{t+1}} = A_t K_t^\alpha L_t^{1-\alpha} + (1 - \delta) K_t, \forall t \geq 1$$  \hspace{1cm} (6)
The equilibrium of the economy

The resource constraint

- The market clearing condition for the goods market was shown to be:

\[ N_{cyt} + N_{cot} + N_{st+1} = A_t K_t^\alpha L_t^{1-\alpha} + (1 - \delta) K_t, \forall t \geq 1, \]

- Recall that the savings of the young constitute the capital stock of the next period:

\[ N_{st+1} = K_{t+1} \text{ for } t \geq 1. \tag{7} \]

Hence, the market clearing condition can be rewritten as

\[ N_{cyt} + N_{cot} + K_{t+1} = A_t K_t^\alpha L_t^{1-\alpha} + (1 - \delta) K_t, \text{ for } t \geq 1. \tag{8} \]
Definition of equilibrium

We are now in a position to define the equilibrium:

- Given any sequence of parameters \( \{A_t\}_{t=1}^\infty \), an equilibrium consists of a sequence of prices \( \{w_t, r_t\}_{t=1}^\infty \), and assignments of the households \( \{c_{yt}, c_{ot+1}, s_{t+1}\}_{t=1}^\infty \) and firms \( \{K_t, L_t\}_{t=1}^\infty \) such that:
  - Given \( \{w_t, r_t\}_{t=1}^\infty \), the households of all generations \( t \geq 1 \) maximize their expected utility so that \( \{c_{yt}, c_{ot+1}, s_{t+1}\} \) satisfy conditions (3), (4) and (5).
  - Given \( \{w_t, r_t\}_{t=1}^\infty \), the firms maximize their profits in all periods, i.e. \( K_t \) and \( L_t \) satisfy conditions (1) and (1) \( \forall t \geq 1 \).
  - All markets clear.
Market clearing conditions

- Labor market:
  \[ L_t = N. \]

- The goods market:
  \[ Nc_{yt} + Nc_{ot} + Ns_{t+1} = A_t K_t^\alpha L_t^{1-\alpha} + (1 - \delta) K_t, \]

- The capital market:
  \[ Ns_{t-1} = K_t. \]
Labor market equilibrium

- Notice that the equilibrium in the labor market is trivial. The agents do not value leisure and so they are willing to work “full time” for any positive salary and in equilibrium $L_t = N \forall t \geq 1$.
- Furthermore, we know that $w_t$ is simply the MPL in period $t$:

$$w_t = (1 - \alpha) A_t K^\alpha_t L_t^{1-\alpha} = (1 - \alpha) A_t \left(\frac{K_t}{N}\right)^\alpha.$$

which depends positively on the capital-labor ratio $\kappa_t = K_t / N$. 
Equilibrium consumption of the young and old

The solution for the wage rates allows us immediately to calculate the consumption levels of the young (generation \( t \)) and old (generation \( t - 1 \)) in \( t \):

\[
c_{yt} = \frac{1}{1 + \beta} w_t = \frac{1}{1 + \beta} (1 - \alpha) A_t \left( \frac{K_t}{N} \right)^\alpha
\]

\[
c_{ot} = (1 + r_t - \delta) \frac{\beta}{1 + \beta} w_{t-1} = \frac{\beta (1 + r_t - \delta)}{1 + \beta} (1 - \alpha) A_t \left( \frac{K_{t-1}}{N} \right)^\alpha.
\]
Goods market equilibrium

- As shown before, the market clearing condition for the goods market is

$$N_{c_yt} + N_{c_{ot}} + N_{s_{t+1}} = A_t K_t^\alpha L_t^{1-\alpha} + (1 - \delta) K_t.$$

- Aggregate consumption in $t$ is equal to $C_t = N_{c_yt} + N_{c_{ot}}$. Similarly, since $N_{s_{t+1}} = K_{t+1}$, we can define gross investment as $I_t = K_{t+1} - (1 - \delta) K_t$:

$$\underbrace{N_{c_yt} + N_{c_{ot}} + K_{t+1} - (1 - \delta) K_t} = \underbrace{C_t = I_t} = \underbrace{A_t K_t^\alpha L_t^{1-\alpha} = Y_t}.$$

Equilibrium allocations

- Notice that $C_t = Y_t - l_T$ and so

$$C_t = A_t K_t^\alpha L_t^{1-\alpha} - K_{t+1} + (1 - \delta) K_t$$

$$= A_t K_t^\alpha N^{1-\alpha} - N \frac{\beta}{1 + \beta} w_t + (1 - \delta) K_t$$

$$= A_t K_t^\alpha N^{1-\alpha} - N \frac{\beta}{1 + \beta} (1 - \alpha) A_t \left( \frac{K_t}{N} \right)^\alpha + (1 - \delta) K_t$$

$$= \left(1 - \frac{\beta}{1 + \beta} (1 - \alpha)\right) A_t K_t^\alpha N^{1-\alpha} + (1 - \delta) K_t$$

$$= \left(1 - \frac{\beta}{1 + \beta} (1 - \alpha)\right) Y_t + (1 - \delta) K_t$$
Equilibrium allocations

- Similarly, $I_t$ is given by

$$I_t = Y_t - C_t$$

$$= A_t K_t^\alpha N^{1-\alpha} - \left(1 - \frac{\beta}{1 + \beta} (1 - \alpha)\right) A_t K_t^\alpha N^{1-\alpha} - (1 - \delta) K_t$$

$$= \frac{\beta}{1 + \beta} (1 - \alpha) A_t K_t^\alpha N^{1-\alpha} - (1 - \delta) K_t.$$
Convergence

- Our objective is to analyze convergence to steady state in an economy in which $A_t = 1$.
- In this economy we know that $s_{t+1}$, $w_t$ and $K_{t+1}$ satisfy:

$$s_{t+1} = \frac{\beta}{1 + \beta} w_t,$$

$$w_t = (1 - \alpha) \left(\frac{K_t}{N}\right)^\alpha,$$

$$K_{t+1} = N \frac{\beta}{1 + \beta} (1 - \alpha) \left(\frac{K_t}{N}\right)^\alpha = \frac{\beta}{1 + \beta} (1 - \alpha) N^{1-\alpha} K_t^\alpha,$$  \hspace{1cm} (9)

- Condition (9) defines the Law of Motion for the capital stock.
Steady state equilibrium

- A steady state equilibrium is a situation in which the equilibrium allocation is the same in every period.

- In other words, if the capital stock at the start of a period $t$ is equal to $\bar{K}$ then the equilibrium must be such that the young in period $t$ decide to save exactly a quantity $Ns_{t+1} = \bar{K}$.

- In the case of our benchmark economy we can demonstrate that the equilibrium converges uniformly to a unique steady state.

- Intuitively, for any economy with $K_t < \bar{K}$ we find that $Ns_{t+1} = K_{t+1} > K_t$. Conversely for any $K_t > \bar{K}$ the young agents will choose a lower capital stock and so $K_{t+1} < K_t$. 
Convergence to steady state

Inspection of the law of motion for capital

\[ K_{t+1} = N \frac{\beta}{1 + \beta} (1 - \alpha) \left( \frac{K_t}{N} \right)^{\alpha} = \frac{\beta}{1 + \beta} (1 - \alpha) N^{1-\alpha} K_t^\alpha, \]

reveals that the right-hand side is a strictly concave function of \( K_t \).

In fact the pair \((A_t, K_t)\) completely defines the outcome of the economy in this period, and in the absence of shocks it also defines the **state of the economy** in period \( t + 1 \). Formally, state variables are pre-determined variables.
Convergence

Notice that the law of motion is given by

\[ K_{t+1} = N \frac{\beta}{1 + \beta} (1 - \alpha) \left( \frac{K_t}{N} \right)^\alpha = \frac{\beta}{1 + \beta} (1 - \alpha) N^{1-\alpha} K_t^\alpha, \]

Hence, the slope of this expression is equal to:

\[ \frac{\partial K_{t+1}}{\partial K_t} = \alpha \frac{\beta}{1 + \beta} (1 - \alpha) N^{1-\alpha} K_t^{\alpha-1} \]

\[ = \alpha \frac{\beta}{1 + \beta} (1 - \alpha) \left( \frac{N}{K_t} \right)^{1-\alpha} \]

This is a continuous function of \( K_{t+1} = g(K_t) \) with

\[ \lim_{K_t \to 0} \left( \frac{\partial K_{t+1}}{\partial K_t} \right) = \infty, \lim_{K_t \to \infty} \left( \frac{\partial K_{t+1}}{\partial K_t} \right) = 0 \] and \( g(0) = 0 \). These features guarantee convergence.
Convergence
Calculation of the steady state capital stock

In steady state $K_{t+1} = K_t = K$. Substituting this equality in our law of motion

$$K_{t+1} = N \frac{\beta}{1 + \beta} (1 - \alpha) \left( \frac{K_t}{N} \right)^\alpha = \frac{\beta}{1 + \beta} (1 - \alpha) N^{1-\alpha} K_t^\alpha,$$

yields

$$K = \frac{\beta}{1 + \beta} (1 - \alpha) N^{1-\alpha} K^\alpha.$$

$$K^{1-\alpha} = \frac{\beta}{1 + \beta} (1 - \alpha) N^{1-\alpha}$$

$$K = \left( \frac{\beta}{1 + \beta} (1 - \alpha) \right)^{\frac{1}{1-\alpha}} N$$

$$\frac{K}{N} = \left( \frac{\beta}{1 + \beta} (1 - \alpha) \right)^{\frac{1}{1-\alpha}}$$
Properties of the steady state

- Economies with the same fundamentals \((\alpha, \beta)\) converge to the same steady state.

- The steady state capital stock is strictly increasing in the constant average savings ratio \(\frac{\beta}{1+\beta} (1 - \alpha)\). The same is not true for steady state consumption (of the young):
  - The young obtain a fraction \((1 - \alpha)\) of national income which is the labor share.
  - They save a fraction \(\frac{\beta}{1+\beta}\) of their income.

- A permanent increase in the savings rate leads to a temporary growth in output and a permanent increase in the steady state level of \(K\).

- Permanent growth in living standards (per capita consumption) requires a permanent increase in \(A\).
A quick comparison with the Solow growth model

The process of capital accumulation and convergence in our OLG model is similar to the mechanics of the Solow growth model. However, there are relevant differences:

- In the Solow model, there are infinitely-lived agents who save an arbitrary fraction of their income.
- In our model the agents have finite lifetimes and they save an optimal fraction of labor income.
  - The fact that the old consume their capital is isomorphic to a 100% depreciation rate.
  - This assumption places a very strict upper-limit on the capital stock,
A first glance at the response to TFP shocks

- To obtain a rough idea about the response of consumption to TFP shocks, we can calculate the elasticity of $C_t$ w.r.t. $A_t$. First notice that:

$$\frac{\partial Y_t}{Y_t} = \frac{\partial Y_t}{\partial A_t} \frac{A_t}{Y_t} = K_t^\alpha L_t^{1-\alpha} \frac{A_t}{Y_t} = 1,$$

- Next,

$$C_t = \left[1 - \frac{\beta}{1+\beta} (1-\alpha)\right] Y_t + (1-\delta) K_t$$

$$\frac{\partial C_t}{C_t} \frac{\partial A_t}{A_t} = \frac{\left[\left(1 - \frac{\beta}{1+\beta} (1-\alpha)\right) K_t^\alpha N^{1-\alpha}\right] A_t}{\left(1 - \frac{\beta}{1+\beta} (1-\alpha)\right) A_t K_t^\alpha N^{1-\alpha} + (1-\delta) K_t} < 1$$
A first glance at the effects of TFP shocks

- Similarly, in the case of investments we obtain

\[ I_t = \frac{\beta}{1 + \beta} (1 - \alpha) Y_t - (1 - \delta) K_t \]

\[
\frac{\partial I_t}{\partial A_t} \frac{I_t}{A_t} = \frac{\beta}{1 + \beta} \left( 1 - \alpha \right) K_t^{\alpha} N^{1-\alpha} \cdot \frac{A_t}{\frac{\beta}{1+\beta} \left( 1 - \alpha \right) A_t K_t^{\alpha} N^{1-\alpha} - (1 - \delta) K_t} > 1
\]

- In sum, in response to a TFP shock the ratio between the percentage changes in \( C \) and \( Y \) is smaller than 1, while investments fluctuate relatively more than output.

- The above features mimic regularities of actual business cycle fluctuations.
Temporary TFP shocks

- In the next slides we will analyze the effects of a temporary shock to TFP in an economy that is initially in steady state.
- In the first step we will calculate the deterministic steady state in which \( A_t = 1 \) for all \( t \).
- Then we consider the effects of an unanticipated increase in \( A \) that lasts for one period.
- Our objective is to show that this shock has persistent effects: the economy does not immediately return to steady state.
The baseline

- Suppose that $\beta = 1$, $\alpha = 0.3$, $\delta = 0.1$, and $N = 100$. In this economy, the steady state capital stock is given by:

$$K = \left(\frac{\beta}{1+\beta} (1 - \alpha)\right)^{\frac{1}{1-\alpha}} N$$

$$= \left(\frac{1}{2}(1 - 0.3)\right)^{\frac{1}{1-0.3}} 100 = 22.319$$

- The steady-state level of output is:

$$Y = K^\alpha L^{1-\alpha} = (22.319)^{0.3} (100)^{1-0.3} = 63.768$$

- Aggregate consumption in steady state is

$$C = \left(1 - \frac{\beta}{1+\beta} (1 - \alpha)\right) K^\alpha N^{1-\alpha} + (1 - \delta)K$$

$$= \left(1 - \frac{1}{2}(1 - 0.3)\right) (22.319)^{0.3} (100)^{1-0.3} + (1 - 0.1)22.319$$

$$= 61.536$$
The steady state level of gross investment is:

\[ I = \frac{\beta}{1 + \beta} (1 - \alpha) A_t K_t^\alpha N^{1-\alpha} - (1 - \delta) K_t \]

\[ = \frac{1}{2} (1 - 0.3) (22.319)^{0.3} (100)^{1-0.3} - (1 - 0.1)22.319 \]

\[ = 2.2319 \]

Notice that in steady state

\[ I = \delta K = (0.1)22.319 = 2.2319, \]

Investment serves entirely to compensate depreciation.
An unanticipated transitory TFP shock

- We now consider the effects of an unanticipated TFP shock. In a given period $t$, the level of $A_t$ jumps up to 1.05.

- In $t + 1$, the value of $A$ returns to its initial value of 1 and stays there forever.

- The fact that the shock is unanticipated rules out any possible anticipation effects.

- So, $K_t = K = 22.319$ and the effect of the TFP shock is a proportional increase in $Y_t$:

$$Y_t = A_t K_t^\alpha N^{1-\alpha}$$

$$= (1.05) (22.319)^{0.3} (100)^{1-0.3} = 66.956$$
The response of consumption

- The higher output leads to an increase in aggregate consumption:

\[
C_t = \left(1 - \frac{\beta}{1 + \beta} (1 - \alpha)\right) A_t K_t^\alpha N^{1-\alpha} + (1 - \delta) K_t
\]

\[
= \left(1 - \frac{1}{2} (1 - 0.3)\right) (1.05) (22.319)^{0.3} (100)^{1-0.3} + (1 - 0.1) 22.319
\]

\[
= 63.609
\]

- Notice that the total amount of resources and hence consumption increase less than proportionally because \(K_t\) is pre-determined.
The response of investment

- The higher level of income also leads to higher investments as the young raise their savings

\[ I_t = \frac{\beta}{1 + \beta} (1 - \alpha) A_t K_t^\alpha N^{1-\alpha} - (1 - \delta) K_t \]

\[ = \frac{1}{2} (1 - 0.3)(1.05)(22.319)^{0.3}(100)^{1-0.3} - (1 - 0.1)22.319 \]

\[ = 3.3476 \]

- Notice that \( I \) increase by \( 100 \times (3.3476 - 2.2319)/2.2319 = 50\%! \)

- Intuition: in steady state \( I = 0.1K \). But due to the TFP shock \( w_t, s_{t+1} \) and \( K_{t+1} \) all increase by 5\%. 
The reponse of the capital stock

- Next period’s capital stock, $K_{t+1}$, is equal to

$$K_{t+1} = (1 - \delta)K_t + I_t$$
$$= (1 - 0.1)22.319 + 3.3476 = 23.435$$

- Consequently, $Y_{t+1} > Y$ although $A_{t+1} = 1$:

$$Y_{t+1} = A_{t+1}K_{t+1}^\alpha L^{1-\alpha}$$
$$= 1 (23.435)^{0.3} (100)^{1-0.3} = 64.708$$

Note que incluso si el shock dura un sólo período, $Y_{t+1}$ es todavía mayor que $Y = 63.768$. 
The fact that $Y_{t+1}$ and $K_{t+1}$ stay above their steady state values, implies that $C_{t+1}$ also stays above the level of $C$:

$$C_{t+1} = \left(1 - \frac{\beta}{1 + \beta}(1 - \alpha)\right) Y_{t+1} + (1 - \delta) K_{t+1}$$

Can we say the same thing about $I_{t+1}$?

$$I_{t+1} = \frac{\beta}{1 + \beta}(1 - \alpha) Y_{t+1} - (1 - \delta) K_{t+1}.$$ 

That’s not clear. On the one hand, $Y_{t+1}$ is high, and so the young can save a high amount. But on the other hand, $K_{t+1}$ is also high and this reduces the need to invest.
Temporary TFP shocks

- The figure on the next slide shows the impulse responses — i.e. the percentage deviations of $C_t$, $I_t$, $K_t$, $Y_t$ from their steady state values in the periods after the shock.

- For example, the first-period deviation in $Y_t$ is defined as:

$$\left( \frac{Y_1 - Y}{Y} \right) \times 100 = \left( \frac{66.956 - 63.768}{63.768} \right) \times 100 = 4.9\%$$

The rest of the deviations is defined in a similar manner.
Deviations

Percentage deviations from steady state values
Summary

- Consumption is shown to be less responsive to TFP shocks than investment (when measured in perc. deviations from steady state). This is consistent with the data.
- The effects of the shock lasts for several periods (persistence). This is due to inter-temporal substitution.
- The agents do not want to consume the entire increase in income in period $t$. As a result, savings and investment increase.
- The increase in investment leads to a rise in the capital stock and this (stock) variable returns sluggishly to its steady state value.
Real business cycle theory

- Our OLG-model produces an expansion period after a one-period TFP shock.
- It is easy to show that a negative TFP shock would lead to a recessionary period in which consumption, income and output would remain below their steady state values for several periods.
- Recurrent shocks would lead to fluctuations around steady state.
- Due to the optimal response of the agents the effect of the shocks propagate through the economy and may persist after the shock has disappeared.
- This propagation and amplification of real shocks is the focus of the so-called real business cycle theory. The inter-temporal substitution of labor and consumption places a prime role in these theories.
Modern business cycle theory

Nowadays all modern business cycle models use setups with rational, forward-looking and maximizing agents.

Contributions to the RBC literature assume perfectly flexible prices and wages and analyze the role of real shocks (and frictions).

Modern neo-Keynesian models use the same setup but assume rigid prices and/or wages. The agents take these frictions in price setting into account when they take their decisions.

Both strands of the business cycle literature do have radically different policy implications.
Questions

In the standard RBC-model labor supply is endogenous and equilibrium hours of work fall in recessions and rise in booms.

- Can we interpret this drop in hours as unemployment?
- Should the government intervene to avoid the fluctuations in labor supply and consumption?