1 Consider an economy populated by a large number $N$ of identical agents with the following preferences over consumption and leisure:

$$U(c, 1 - l) = \frac{c^{1-\sigma}}{1-\sigma} + \frac{(1 - l)^{1-\sigma}}{1-\sigma},$$

where $c$ denotes consumption, $1 - l$ is leisure and $l$ is the share of time devoted to work.

a. Derive the expression for the marginal rate of substitution between consumption and leisure $MRS_{c,l} = dc/dl$ (5 points).

In autarky, each agent is able to produce $y = Al$ units of the unique final good using labor as the only input. The parameter $A > 0$ measures the state of technology.

b. Derive the optimal solution for $l^*$ assuming that the agent maximizes her utility subject to the resource constraint $c = Al$ (15 points).

Now assume there is a representative firm that hires labor in a competitive labor market taking the wage rate, $w$, as given. The objective of the firm is to maximize its profits $\Pi = AL^f - wL^f$, where $L^f$ is (aggregate) labor demand of the representative firm.

c. Solve the maximization problem of the representative firm and a representative agent and demonstrate that the value of $l^*$ is the same as in autarky (15 points).

d. Illustrate the equilibrium in the labor market with a figure that contains the schedules for aggregate labor demand and supply as a function of $w$ (5 points).

e. Use your answers in c and d to analyze the effects of the introduction of a proportional income tax $t = \tau w$ that is paid by the workers (10 points).
2. Consider a pure exchange economy of two periods. All agents have the following preferences over current and future consumption, \( c_1 \) and \( c_2 \) respectively:

\[
U(c_1, c_2) = \sqrt{c_1} + \beta \sqrt{c_2},
\]

with \( 0 < \beta \leq 1 \). The \( N_a \) agents of type \( a \) receive a known endowment of 2 units of the final good in period 1 and 0 in period 2, i.e. \( y^a_1 = 1 \) and \( y^a_2 = 0 \). Similarly, the \( N_b \) agents of type \( b \) receive an endowment stream of \( y^b_1 = 0 \) and \( y^b_2 = 1 \). The agents have access to a perfectly competitive credit market and we denote the real interest rate by \( R \).

a. Derive the expression for the Inter-temporal Marginal Rate of Substitution \( dc_1/dc_1 \) (5 points).

b. Solve the optimization problem of a representative agent for arbitrary values of \( R, \beta \) and the present value of lifetime resources, \( x \). Your answer should include the expressions for the optimal consumption levels \( c_1^*, c_2^* \) and bond holdings \( b_1^* \) (or savings) (20 points).

c. Write down the equilibrium condition for the credit market and derive the equilibrium interest rate \( R \) for the case in which \( N_b = 2N_a \) and \( \beta = 1 \). How much do the agents consume in both periods? (20 points)

d. Derive the necessary condition for an equilibrium in which \( \beta(1+R) = 1 \) and all agents choose a constant consumption profile (5 points).