The Labour-Leisure Choice
Part 1: Robinson Crusoe

Dynamic Macroeconomic Analysis

Universidad Autónoma de Madrid

Autumn 2012
The Economics of Robinson Crusoe

- Suppose the economy is populated by a single individual (Robinson Crusoe). At the end of each day, Robinson consumes the proceeds of his work (coco nuts).

- To characterize Robinson’s choices, we need to specify his:
  - Production technology
  - Preferences over consumption and leisure (or work)
  - Time constraint

- For the moment we assume that the unique final good is perishable. Hence, Robinson cannot save — *i.e.* move goods to the next period.
Production technology

- The production technology is fully described by a *production function*. It specifies a unique level of output for each combination of inputs.

- Formally,

\[ y = Af(k, l) \]

where \( k \) denotes capital (machines) and \( l \) denotes the units of labor (time devoted to work). Finally, \( A \) denotes the level of technology or Total Factor Productivity (TFP).

- Throughout the course we often consider production functions of the type:

\[ y = A l^\alpha k^{1-\alpha} \text{ para } \alpha \in (0, 1) \]
Cobb-Douglas Production Functions

Figure 2.1: Cobb-Douglas Production
Neo-classical production functions

The class of neo-classical production functions satisfies the following characteristics:

1. The production function passes through the origin \((F(0, 0) = 0)\)

2. Strictly positive and decreasing marginal productivity

   i. \(F_k(k, l) = \frac{\partial F(k, l)}{\partial k} > 0; \quad F_l(k, l) = \frac{\partial F(k, l)}{\partial l} > 0\)

   ii. \(F_{kk}(k, l) = \frac{\partial^2 F_k(k, l)}{\partial k^2} < 0; \quad F_{ll}(k, l) = \frac{\partial^2 F_l(k, l)}{\partial l^2} < 0\)

3. Constant returns to scale \((F(\lambda k, \lambda l) = \lambda F(k, l))\)
Robinson’s Production Technology

- Suppose that \( k = 1 \). In this case we can write the per capita output of Robinson, \( y \), as a function of \( l \),

\[ y = f (l) = Al^\alpha \]

- Similarly, Robinson’s Marginal Product of Labour (MPL) is defined as:

\[ PML = f' (l) = \frac{\partial y}{\partial l} \]

- With our Cobb-Douglas production function we obtain:

\[ f' (l) = \frac{\partial y}{\partial l} = \alpha Al^{\alpha-1} > 0 \]

\[ f'' (l) = \frac{\partial^2 y}{\partial l^2} = \alpha (\alpha - 1) Al^{\alpha-2} < 0 \]
Pendiente = \frac{\text{y}_t - \text{y}_{t-1}}{\text{l}_t - \text{l}_{t-1}} = f'(l) = \frac{\text{PML}}{\text{l}_t - \text{l}_{t-1}}
Preferences

- We represent the preferences with the aid of a (twice continuously differentiable) utility function

\[ u(c, l) \]

with

\[ u_1(c, l) = \frac{\partial u}{\partial c} > 0 \quad \text{and} \quad u_2(c, l) = \frac{\partial u}{\partial l} < 0. \]

- Assuming that agents have one unit of time at their disposal we can also use the following notation:

\[ u(c, 1 - l), \]

where \( 1 - l \) represents leisure and in this case \( u_2 > 0 \).
Example: Logarithmic Utility

Throughout the course we frequently use the convenient example of logarithmic utility. An example of this class of preferences is:

$$u(c, 1-l) = \log(c) + \rho \log(1-l),$$

where $\rho > 0$ is a parameter that measures the relative weight of leisure in the utility function.
Indifference curves

Indifference curves are defined by all the combinations of $c$ and $l$ that provide the same level of utility

- Formally, suppose we are interested in the combinations of $c$ and $l$ that yield $\bar{u}$ utils. In other words, with $\rho = 1$

  $$\bar{u} = \log(c) + \log(1 - l)$$

- Notice that we can rewrite this expression as

  $$e^{\bar{u}} = c(1 - l)$$

  and so

  $$c = \frac{e^{\bar{u}}}{1 - l}$$

- The indifference curves have a positive (negative) slope in the plane $(c, l) (\bar{c}, l(1 - l))$. 
Indifference curves
The Marginal Rate of Substitution between Consumption and Leisure

The slope of the indifference curve is known as the MRS between consumption and leisure.

Suppose we raise \( l \) by an infinitesimally small amount \((dl)\). The MRS tells us by how much we need to raise \( c \) to keep the level of utility constant. Taking the total derivative,

\[
d\bar{u} = 0 = u_1(c, 1-l) \, dc + u_2(c, 1-l) \cdot (-1) \, dl
\]

\[
\underbrace{u_1(c, 1-l) \, dc} = \underbrace{u_2(c, 1-l) \, dl}
\]

utility gain

utility loss

\[
MRS = \frac{dc}{dl} = \frac{u_2(c, 1-l)}{u_1(c, 1-l)} > 0
\]
The Labour-Leisure Choice

We now have all the necessary elements to characterize the optimal choice of labor and leisure:

- Recall that the unique final good is perishable.
- Given that Crusoe cannot store goods, in every period he will consume his entire output. Hence,

  \[ c_t = y_t = f(l_t) \quad \forall t \]

- In the absence of shocks Robinson therefore needs to resolve the same static problem in each period.
Formally, Robinson needs to solve the following constrained optimization problem

\[
\max_{c, l} [u(c, 1 - l)]
\]

s.a

\[c = f(l)\]

The best option is to solve this problem with the help of the substitution method. Replacing \(c\) in the maximand by \(f(l)\), we obtain:

\[
\max_{l} [u(f(l), 1 - l)]
\]
The optimal choice

- The first-order condition (FOC) that characterizes the optimal choice is given by:

\[ u_1 (f (l), 1 - l) f' (l) + u_2 (f (l), 1 - l) (-1) = 0 \]

\[ \frac{u_1(f(l), 1 - l)f'(l)}{u_1(c, 1 - l)} = \frac{u_2(f(l), 1 - l)}{u_1(c, 1 - l)} \]

\[ MRS = \frac{u_2(c, 1 - l)}{u_1(c, 1 - l)} = f'(l) = MPL \]

Notice that MPL can be interpreted as the marginal rate of transformation of labor in goods.
Graphical representation

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Exercise I

Suppose Robinson’s preferences satisfy

\[ u(c, 1 - l) = \ln(c) + \rho \ln(1 - l). \]

a. Derive the expression for the MRS.

b. How does the value of the MRS change if we raise the value of \( l \)?

c. What is the effect on MRS of a rise in the value of \( \rho \)?
Exercise I

Suppose Robinson’s preferences satisfy

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(a) Derive the expression for the MRS.

(b) How does the value of the MRS change if we raise the value of \( l \)?

(c) What is the effect on MRS of a rise in the value of \( \rho \)?

**Answers:**

(a) \( MRS = \rho \left( \frac{c}{1 - l} \right) \).

(b) The MRS is an increasing function of \( l \). At higher values of \( l \) the worker needs a larger compensation for further losses of leisure.

(c) A rise in \( \rho \) leads to a rise in the value of the RMS for given values of \( c \) and \( l \).
Exercise II

Suppose that

\[ f(l) = Al^\alpha, \]  
and that \( u(c, 1 - l) = \log(c) + \log(1 - l). \)

a. Derive the FOC and demonstrate that the optimal values of \( c^* \) and \( l^* \) satisfy the following condition:

\[ \frac{u_2(c^*, 1 - l^*)}{u_1(c^*, 1 - l^*)} = f'(l^*) \]

b. Derive the expressions for \( c^* \) and \( l^* \) and analyze the effects of an increase in \( A \) and \( \alpha \) on the optimal choices of Robinson.
Solutions

a. The FOC associated with \( \max \left[ \log(A l^\alpha) + \log(1 - l) \right] \) is given by:

\[
\frac{1}{A l^\alpha} A l^\alpha - 1 = \frac{1}{1 - l}
\]

Hence:

\[
MPL = f'(l) = \alpha A l^{\alpha - 1} = \frac{A l^\alpha}{1 - l} = MRS
\]

b. The FOC implies that

\[
\alpha = \frac{l}{1 - l}
\]

Thus, \( l^* = \frac{\alpha}{1 + \alpha} \) and \( c^* = A \left( \frac{\alpha}{1 + \alpha} \right)^\alpha \).

It follows immediately that \( \partial l^*/\partial \alpha > 0 \) and \( \partial l^*/\partial A = 0 \), while \( \partial c^*/\partial \alpha > 0 \) and \( \partial c^*/\partial A > 0 \).
The Lagrange method

The alternative is to use the Lagrange method. The Lagrangian associated with our optimization problem can be written as

\[ L = u(c, 1 - l) + \lambda [f(l) - c] \]

where \( \lambda \) denotes the Lagrange multiplier. The FOCs are given by:

\[
\frac{\partial L}{\partial c_t} = u_1(c, 1 - l) - \lambda = 0
\]

\[
\frac{\partial L}{\partial l_t} = u_2(c, 1 - l)(-1) + \lambda f'(l_t) = 0
\]

\[
\frac{\partial L}{\partial \lambda} = f(l_t) - c_t = 0
\]

As \( \lambda = u_1(c, 1 - l) \), we again obtain the standard optimality condition

\[
\frac{u_2(c, 1 - l)}{u_1(c, 1 - l)} = f'(l)
\]
Productivity shocks

Our next objective is to study the effects of productivity shocks on the choices of Robinson.

In general, productivity shocks produce both income and substitution effects.

In the case Robinson’s island economy, productivity shocks may be due changes in the weather conditions. In modern business cycle models on the contrary productivity shocks may be driven by a wide range of factors such as changes in energy prices, climate conditions, arrival of new technologies etc.

For the moment we maintain the assumption of perishable goods. The optimization problem of Robinson is therefore static although productivity shocks may be persistent.
Example I: Parallel shift of the production function
Example II: A positive TFP shock
Definitions

**Income effect:** An increase in output for any given choice of $l$, but no change in the MPL schedule.

**Substitution effect:** A change in the MPL, or equivalently the marginal rate of transformation of labor into consumption.
Example III: A Pure Income Effect

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Example IV: The Effects of a Pure Substitution Effect

\[
y = f(l)
\]
Combined Income and Substitution Effects

Suppose that Robinson has a fixed endowment $a \geq 0$ of consumption goods and that $f(l) = AL$. Accordingly,

$$c = a + Al$$

and

$$\max_l \left[ \log (a + Al) + \log (1 - l) \right]$$

Notice that the FOC for this problem can be written as

$$\frac{1}{a + Al} A = \frac{1}{1 - l}$$

And so,

$$l^* = \frac{A - a}{2A}$$

and

$$c^* = \frac{A}{2} + \frac{a}{2}$$
Combined Income and Substitution Effects

In our example,

- **The increase in** $a$ generates a pure income effect. In response to this positive income effect, Robinson increases both leisure and consumption.

- **The increase in** $A$ produces both income and substitution effects. The latter dominate as Robinson’s optimal labor supply is rising in $A$. 
Disentangling Income and Substitution Effects

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