The Labour-Leisure Choice
Part II: A Competitive Labor Market

Dynamic Macroeconomic Analysis

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Labor-leisure choices in a competitive economy

- Suppose now that the economy is populated by many identical agents.
- Each agent maximizes his or her own utility taking prices as given.
- In our simple example there are just two markets for labour services and capital (machines).
- There are also many identical firms who hire labor and capital. The production function of firms is given by:
  \[ y = f(l, k) = Al^\alpha k^{1-\alpha} \text{ para } \alpha \in (0, 1) \]
  
- The price of the final good is normalized to one, while the wage rate and the rental price of capital are denoted by \( w \) and \( r \), respectively.
The representative firm

With constant returns to scale the firm-size distribution of firms is irrelevant (see problem set 1). In equilibrium, all firms opt for the same capital-labour ratio and total output does not depend on the number of firms.

Thus, without loss of generality we can assume the existence of a single representative firm.

The representative firm acts as a price taker and in equilibrium it hires the entire stock of capital, $K$, and labor, $L$. 
The problem of the representative firm

The problem of the representative firm is given by:

\[
\max_{L_f, K_f} \pi = A (L_f)^\alpha K_f^{1-\alpha} - wL_f - rK_f
\]

where \( L_f \) (\( K_f \)) denotes the firm’s demand for labor (capital).

The two FOCs for capital and labor are given by:

\[
\begin{align*}
(1 - \alpha) AK_f^{-\alpha} (L_f)^\alpha &= r \\
\alpha AK_f^{-\alpha} (L_f)^{\alpha-1} &= w
\end{align*}
\]
The problem of a representative individual

Like before we assume that each individual is endowed with one unit of capital (a machine) and time. But this time we assume that workers cannot operate their own machines.

Hence, each individual faces the following problem:

$$\max_{c, l} \left[ \log(c) + \log(1 - l) \right],$$

subject to

$$c = wl + r \quad \text{(total income)}.$$ 

Hence, each individual obtains income from renting work, $wl$, and capital, $r$, to firms. Notice that the supply of capital is totally inelastic.
The solution

After substituting the budget constraint into the objective function, we obtain the following problem:

$$\max_l \left[ \log(wl + r) + \log(1 - l) \right].$$

The FOC is given by

$$\frac{1}{wl + r} w = \frac{1}{1 - l}.$$  

which can be solved for the optimal solutions

$$l^* = \frac{w - r}{2w} = \frac{1}{2} - \frac{r}{2w}$$ and $$c^* = \frac{w + r}{2}.$$
The individual problem

\[ c = w l + r \]

\[ u_1 \]

\[ u_2 \]

\[ c^* \]

\[ r \]

\[ w l \]

\[ r \]

\[ l^* \]

\[ A \]

\[ \text{consumo} \]

\[ \text{trabajo} l \]
**Equilibrium**

We are now in a position to derive the equilibrium and to demonstrate the equivalence between Robinson’s choices in autarky and in the market equilibrium.

- Suppose there is a total of $N$ Robinsons in the economy.
- The total or aggregate supply of capital is $K_s = N$.
- Aggregate labor supply is $L_s = Nl^*$. 
- In *equilibrium* all agents solve their individual problem taking prices as given and we have two market-clearing conditions:

\[
K_f = K_s \quad \text{(capital market equilibrium)}
\]

\[
L_f = L_s \quad \text{(labor market equilibrium)}
\]
Equilibrium

Let’s start with the case of the representative firm:

- In equilibrium, the output of the firm is given by
  \[ Y_f = A(K_f^{1-\alpha} L_f^\alpha) = A(N^{1-\alpha} (N^l)^\alpha) = AN (l^*)^\alpha. \]
  while the output per worker satisfies:
  \[ y = \frac{Y_f}{N} = A (l^*)^\alpha. \]

- Similarly, the equilibrium factor prices are
  \[ r = (1 - \alpha) A(N)^{-\alpha} (N^l)^\alpha = (1 - \alpha) A (l^*)^\alpha, \]
  \[ w = A\alpha K_f^{1-\alpha} L_f^\alpha = A\alpha (N)^{1-\alpha} (N^l)^\alpha = A\alpha (l^*)^{\alpha-1}. \]
Equilibrium

- In equilibrium the representative firm earns zero profit as

\[
\pi = A(K_f^{1-\alpha} L_f^\alpha) - wL_f - rK_f
\]

\[
= [AN^{1-\alpha} (Nl^*)^\alpha - \alpha A (l^*)^{\alpha-1} (Nl^*) - (1 - \alpha) A (l^*)^\alpha (N)].
\]

\[
= AN (l^*)^\alpha - \alpha AN (l^*)^\alpha - (1 - \alpha) AN (l^*)^\alpha
\]

\[
= 0.
\]

- The equilibrium income of each individual agent is equal to per capita output \(y\):

\[
i = w l^* + r = \]

\[
= \alpha A (l^*)^{\alpha-1} l^* + (1 - \alpha) A (l^*)^\alpha
\]

\[
= \alpha A (l^*)^\alpha + (1 - \alpha) A (l^*)^\alpha
\]

\[
= y.
\]
Equilibrium
Summary

- All markets clear (demand equal supply at the equilibrium prices)
- The relevant (relative) prices are
  \[ r = (1 - \alpha)A(I^*)^\alpha \quad \text{and} \quad w = \alpha A(I^*)^{\alpha - 1}. \]
- Each individual earns an income of \( y \) as
  \[ i = \alpha A(I^*)^\alpha + (1 - \alpha)A(I^*)^\alpha = A(I^*)^\alpha. \]
Equilibrium
Summary

- Per capita consumption is equal to
  \[ c^* = A (l^*)^\alpha. \]

- Optimal individual labor supply is
  \[ l^* = \frac{w - r}{2w}. \]

- Aggregate consumption equals aggregate output
  \[ Y_f = AN (l^*)^\alpha = Nc = NA (l^*)^\alpha. \]

- The representative firm obtains zero profits
It is important to notice that the equilibrium choices of Robinson coincide with the choices Robinson would take in autarky. In the market equilibrium, Robinson’s labor supply decisions are a function of the wage rate $w$. By contrast, in autarky Robinson’s decisions are driven by the value of MPL as he solves

$$\log(c) + \log(1 - l),$$

subject to

$$c = A(l^*)^\alpha.$$

But in equilibrium $w = MPL$. 
The Labor Market

Let’s analyze the labor market in some more detail.

- On the one hand, we found that

\[ L_s = Nl^* = N \left[ \frac{1}{2} - \frac{r}{2w} \right], \]  

(3)

Hence, \( L_s \) is an increasing function of \( w \) (for given values of \( r \)). Does this make sense?

- On the other hand, the optimal demand of labor \( L_f \) of the representative firm is implicitly defined by

\[ A\alpha K_f^{1-\alpha} L_f^{\alpha-1} = w, \]

which implies that

\[ L_f^{1-\alpha} = \frac{A\alpha K_f^{1-\alpha}}{w} \implies L_f = \left( \frac{A\alpha K_f^{1-\alpha}}{w} \right)^{\frac{1}{1-\alpha}} \]  

(4)

In other words, aggregate labor demand is a strictly decreasing function of \( w \).
The Labor Market

The previously defined functions of aggregate labor demand and labor supply give rise to the standard diagram of the classical model of the labor market.
The Capital Market

- Each individual supplies one unit of capital to the market and so the aggregate supply is capital is totally inelastic and equal to \( K_s = N \).
- Aggregate demand for capital is implicitly defined by the corresponding FOC

\[
A(1 - \alpha) K_f^{-\alpha} L_f^\alpha = r.
\]

For given values of \( L_f \)

\[
K_f^\alpha = \left( \frac{A(1 - \alpha) L_f^\alpha}{r} \right)
\]

(5)

\[
K_f = \left( \frac{A(1 - \alpha)}{r} \right)^{\frac{1}{\alpha}} L_f,
\]

is a decreasing function of \( r \).
The Capital Market

The equilibrium in the capital market is trivial:

\[ rs = N \]
A Useful Numerical Example

- In general, it is hard to calculate the exact equilibrium values for \( \{w, r, l, Y, K, L\} \). But sometimes we are lucky as the following example illustrates.
- Suppose \( \alpha = 0.5 \), \( N = 100 \), \( A = 1 \).
- We know that

\[
Y_f = A(K_f^{1-\alpha}L_f^\alpha) = (100^{0.5}(100l^*)^{0.5}) = 100(l^*)^{0.5}.
\]

while the expression for capita output is given by

\[
y = \frac{Y_f}{N} = \frac{100(l^*)^{0.5}}{100} = (l^*)^{0.5}
\]
A Useful Numerical Example

In our example the equilibrium prices are given by:

\[ r = A(1 - \alpha)(N)^{-\alpha}(NL^*)^\alpha = 0.5 (l^*)^{0.5}, \]

and

\[ w = A\alpha K_f^{1-\alpha} L_f^{\alpha-1} = 0.5 (l^*)^{-0.5}. \]

From these two equations it follows that:

\[ \frac{r}{w} = \frac{0.5 (l^*)^{0.5}}{0.5 (l^*)^{-0.5}} = l^* \]
A Useful Numerical Example

- Hence, in any equilibrium it must be true that
  \[ r = wl^*. \]

- Inserting \( r = wl^* \) in our expression for \( l^* \)
  \[ l^* = \frac{w - r}{2w}, \]

we find that
  \[ 2wl^* = w - r = w - wl^* \]

which gives rise to the following solution
  \[ 3wl^* = w \implies l^* = \frac{1}{3}. \]
A Useful Numerical Example

- We now have a complete solution for all the relevant equilibrium quantities and prices:

\[ Y_f = 100 \left(\frac{1}{3}\right)^{0.5} = 57.735. \]

while per capita output is equal to

\[ y = \frac{Y_f}{N} = \frac{100 \left( I^* \right)^{0.5}}{100} = \left( I^* \right)^{0.5} = \left( \frac{1}{3} \right)^{0.5} = 0.57735 \]

- Similarly, the equilibrium factor prices are:

\[ r = A(1 - \alpha)N^{-\alpha}(NL^*)^\alpha = 0.5 \left( I^* \right)^{0.5} = 0.5 \left( \frac{1}{3} \right)^{0.5} = 0.28868, \]

and

\[ w = A\alpha K_f^{1-\alpha} L_f^{\alpha-1} = 0.5 \left( I^* \right)^{-0.5} = 0.5 \left( \frac{1}{3} \right)^{-0.5} = 0.86603. \]
A Useful Numerical Example

- Next,
  \[ c = y = 0.57735, \]
  and per capita income is
  \[ y = w l^* + r = \frac{1}{3} 0.86603 + 0.28868 = 0.57735. \]

- Notice also that
  \[ \frac{w l}{y} = \frac{\frac{1}{3} 0.86603}{0.57735} = 0.5 \quad \frac{r}{y} = \frac{0.28868}{0.57735} = 0.5 \]

Hence, \( \alpha \) is not only the elasticity of the production function w.r.t. \( L \). It also defines what is called the labor share — i.e. the share of total income that accrues to workers. This last object can be measured in the data.