Equilibrium in Two Periods

Dynamic Macroeconomic Analysis

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Roadmap

- So far, we have studied consumption and savings decisions and labor supply decisions in a two-period setting. However, in the case of labor supply, we analyzed a partial equilibrium setting with exogenous wages.

- In this theme we derive the general equilibrium allocations in two periods with endogenous labor supply, savings and consumption.

- Second, we will analyze the response of the equilibrium to transitory and permanent shocks to TFP.

- Our model is still incomplete (we ignore the role of capital and the economy only lasts for two periods), but it neatly illustrates essential features of RBC theory.
The environment

- The economy lasts for two periods.
- There are two groups of agents with different discount rates. All agents value consumption and leisure and need to decide labor supply in both periods. The labor market is competitive.
- At the other side of the labor market there is a representative firm that hires labor services taking the competitive wage in each period as given.
- At the end of the first period the agents can access the credit market to borrow or lend resources.
The Household Problem

- The households choose \((l_1, l_2, c_1, c_2)\) taking prices \((w_1, w_2, R)\) as given. Formally,

\[
\max_{c_1, c_2, l_1, l_2} \left\{ \ln c_1 + \ln(1 - l_1) + \beta \left( \ln c_2 + \ln(1 - l_2) \right) + \lambda \left[ w_1 l_1 + \frac{1}{1 + R} w_2 l_2 - c_1 - \frac{1}{1 + R} c_2 \right] \right\},
\]

where \(\lambda\) is the Lagrange multiplier.

- Notice that we have solved this problem in the previous lecture.
The Household Problem

The first-order conditions

\[ \frac{1}{c_1} - \lambda = 0 \quad (c_1) \]

\[ \beta \frac{1}{c_2} - \lambda \frac{1}{1 + R} = 0 \quad (c_2) \]

\[ - \frac{1}{1 - l_1} + \lambda w_1 = 0 \quad (l_1) \]

\[ -\beta \frac{1}{1 - l_2} + \lambda \frac{w_2}{1 + R} = 0. \quad (l_2) \]
In a first step we derive the consumption Euler condition:

\[ c_2 = \beta (1 + R) c_1, \]

Inserting this condition in the budget constraint we obtain:

\[ (1 + \beta) c_1 = x \implies c_1 = \frac{x}{1 + \beta}, \]  

\[ c_2 = c_1 \beta (1 + R) = \frac{\beta x (1 + R)}{1 + \beta}. \]
Consumption decisions

- In the next step we combine the FOCs for $c_1$ and $l_1$:

$$\frac{1}{1 - l_1} = \frac{1}{c_1} w_1 \implies w_1 l_1 = w_1 - c_1.$$  \hfill (3)

- Similarly, inserting $\lambda = u'(c_1)$ into the FOC for $l_2$

$$\frac{1}{1 - l_2} - \beta \frac{1}{1 - l_2} + \lambda \frac{w_2}{1 + R} = 0.$$   

we obtain the following condition:

$$\beta \frac{1}{1 - l_2} = \frac{1}{c_1} \frac{w_2}{1 + R}.$$

- Hence,

$$c_1 \beta (1 + R) = w_2 - w_2 l_2 \implies \frac{w_2 l_2}{1 + R} = \frac{w_2}{1 + R} - c_1 \beta.$$  \hfill (4)
Consumption decisions

- The value of the agent’s lifetime resources can now be written as

\[ x = w_1 l_1 + \frac{w_2 l_2}{1 + R} = w_1 - c_1 + \frac{w_2}{1 + R} - c_1 \beta. \]

- Given that \( c_1 = \frac{x}{1+\beta} \), we can thus write

\[ c_1 = \frac{w_1 - c_1 + \frac{w_2}{1+R} - c_1 \beta}{1 + \beta}, \]

leading to the following solutions

\[ c_1^* = \frac{1}{2(1 + \beta)} \left[ w_1 + \frac{w_2}{1 + R} \right]. \]

\[ c_2^* = \beta (1 + R) \frac{1}{2(1 + \beta)} \left[ w_1 + \frac{w_2}{1 + R} \right]. \]
Labor supply decisions

Given that we have demonstrated that \( w_1 l_1 = w_1 - c_1 \), we have

\[
w_1 l_1^* = w_1 - c_1^*
\]

\[
= w_1 - \frac{1}{2(1+\beta)} \left[ w_1 + \frac{w_2}{(1+R)} \right]
\]

\[
w_1 l_1 = w_1 \left[ 1 - \frac{1}{2(1+\beta)} \right] - \frac{1}{2(1+\beta)} \frac{w_2}{(1+R)}
\]

\[
l_1^* = \frac{1+2\beta}{2(1+\beta)} - \frac{1}{2(1+\beta)} \frac{1}{1+R} \frac{w_2}{w_1}.
\]
Similarly, our previous results imply that $w_2 l^*_2 = w_2 - c^*_1 \beta (1 + R)$,

$$w_2 l_2 = w_2 - c_1 \beta (1 + R)$$

$$= w_2 - \frac{\beta (1 + R)}{2(1 + \beta)} \left[ w_1 + \frac{w_2}{(1 + R)} \right]$$

$$= w_2 \left[ 1 - \frac{\beta (1 + R)}{2(1 + \beta)} \frac{1}{1 + R} \right] - \frac{\beta (1 + R)}{2(1 + \beta)} w_1.$$

$$l^*_2 = \frac{2 + \beta}{2(1 + \beta)} - \frac{\beta (1 + R)}{2(1 + \beta)} w_1.$$
Summary household decisions

In equilibrium, the household decisions concerning \((c_1, c_2, l_1, l_2)\) satisfy

\[
c_1^* = \frac{1}{2(1 + \beta)} \left[ w_1 + \frac{w_2}{(1 + R)} \right]
\]

\[
c_2^* = \beta(1 + R) \frac{1}{2(1 + \beta)} \left[ w_1 + \frac{w_2}{(1 + R)} \right]
\]

\[
l_1^* = \frac{1 + 2\beta}{2(1 + \beta)} - \frac{1}{2(1 + \beta)} \frac{1}{1 + R} \frac{w_2}{w_1}
\]

\[
l_2^* = \frac{2 + \beta}{2(1 + \beta)} - \frac{\beta(1 + R)}{2(1 + \beta)} \frac{w_1}{w_2}
\]
The representative firm

- Like before we assume that there is a representative firm that acts as a price taker in the labor and the product market.

- For simplicity, we assume that production function is linear in labor:

\[ Y_j^f = A_j L_j^f, \]

where \( A_j \) denotes the value of TFP in period \( j \), \( Y_j^f \) is output in period \( j \), and \( L_j^f \) is the firm’s labor demand in \( j \).
The problem of the representative firm

- In each period the representative firm solves the following static problem:

\[ \pi_j = \max_{L_f} [A_j L_f^j - w_j L_f^j], \]

where \( w_j \) denotes the competitive wage in period \( j \).

- The FOC associated with the firm’s problem implies that

\[ A_j = w_j. \]

- In other words, in each period wages are uniquely determined by the value of TFP, i.e. \( w_1 = A_1 \) and \( w_2 = A_2 \).
To generate a credit market, we will once again assume that there are various groups of agents with different discount rates.

Let's define $\beta_i$ as the discount factor of agent $i$, where $i = 1, \ldots, N$.

Accordingly,

\begin{align*}
    c_{1,i}^* &= \frac{1}{2(1 + \beta_i)} \left[ w_1 + \frac{w_2}{1 + R} \right]. 
    \quad (5) \\
    c_{2,i}^* &= \beta_i (1 + R) \frac{1}{2(1 + \beta_i)} \left[ w_1 + \frac{w_2}{1 + R} \right] 
    \quad (6)
\end{align*}
Recall that
\[ b_{1,i} = w_{1,l_{1,i}} - c_{1,i} \]
Furthermore, we know that \( w_{1,l_{1,i}} = w_{1,i} - c_{1,i} \). Hence,
\[
\begin{align*}
  b_{1,i}^* &= w_{1,l_{1,i}} - c \\
  &= w_{1} - 2c_{1,i} \\
  &= w_{1} - 2 \frac{1}{2(1 + \beta_i)} \left[ w_{1} + \frac{w_{2}}{(1 + R)} \right] \\
  &= w_{1} - \frac{1}{1 + \beta_i} \left[ w_{1} + \frac{w_{2}}{(1 + R)} \right] \\
  &= \frac{w_{1}\beta_i - \frac{w_{2}}{1+R}}{1 + \beta_i}
\end{align*}
\]
Labor supply

Finally, recall that labor supply in both periods satisfies:

\[ l_{1,i}^* = \frac{1 + 2\beta_i}{2(1 + \beta_i)} - \frac{1}{2(1 + \beta_i)} \frac{1}{1 + R} \frac{w_2}{w_1}. \]  

(8)

and in the second period

\[ l_{2,i}^* = \frac{2 + \beta_i}{2(1 + \beta_i)} - \frac{\beta_i(1 + R)}{2(1 + \beta_i) w_1} \frac{w_1}{w_2}. \]  

(9)
Definition of equilibrium

Following standard conventions, the competitive equilibrium in this two-period economy can be defined as follows:

- Given $A_1$, $A_2$ and $\beta_i$ for $i = 1, \ldots, N$, the equilibrium of this economy consists of a set of prices $w_1$, $w_2$, and $R$ and allocations $c_{1,i}^*$, $c_{2,i}^*$, $b_{1,i}^*$, $l_{1,i}^*$, and $l_{2,i}^*$ for the households and $L_1^f$ and $L_2^f$ for the firm.
- Given $w_1$, $w_2$, and $R$, the allocations of the household solve the household problem, i.e. $c_{1,i}^*$, $c_{2,i}^*$, $b_{1,i}^*$, $l_{1,i}^*$, and $l_{2,i}^*$ están dadas por las ecuaciones (5), (6), (7), (8) y (9).
- The firms maximize profits given $A_1$ and $A_2$, i.e.

$$w_1 = A_1 \text{ and } w_2 = A_2.$$  

- All markets clear.
Market clearing conditions

- Credit market:
  \[ \sum_{i}^{N} b_{1,i}^* = 0, \]

- Goods market:
  \[ \sum_{i}^{N} c_{1,i}^* = A_1 L_1^f \text{ and } \sum_{i}^{N} c_{2,i}^* = A_2 L_2^f \]

- Labor market:
  \[ \sum_{i}^{N} l_{1,i}^* = L_1^f \text{ and } \sum_{i}^{N} l_{2,i}^* = L_2^f \]
Walras’ Law

Remember that the final good is perishable. Hence, in every period we expect that the agents consume the entire value of aggregate output.

Notice also that the equilibrium in the labor market is satisfied trivially: the firm is willing to hire any amount of labor at the equilibrium wages $w_j^* = A_j$.

Given the equilibrium in the labor market, we can show that the equilibrium in the goods market implies equilibrium in the credit market, and vice versa.
Proof

• For each agent $i$ the first-period budget constraint can be written as

$$b_{1,i} = w_1 l_{1,i} - c_1.$$ 

• Summing over all agents we get

$$\sum_{i}^N b_{1,i} = \sum_{i}^N w_1 l_{1,i} - \sum_{i}^N c_{1,i} = w_1 \sum_{i}^N l_{1,i} - \sum_{i}^N c_{1,i}.$$ 

• Finally, if the labor market and the goods market are in equilibrium, then aggregate savings satisfies

$$\sum_{i}^N b_{1,i}^* = A_1 \sum_{i}^N l_{1,i}^* - \sum_{i}^N c_{1,i}^*$$

$$= A_1 L_1^f - \sum_{i}^N c_{1,i}^* = 0,$$

Hence, the credit market is also in equilibrium.
Let’s start with period 1.

We have shown that
\[
c_{1,i}^* = \frac{1}{2(1 + \beta_i)} \left[ A_1 + \frac{A_2}{(1 + R)} \right].
\]

Hence, aggregate demand for the final good is equal to
\[
C_1^d = \sum_i^N c_{1,i}^* = \left[ A_1 + \frac{A_2}{(1 + R)} \right] \sum_i^N \frac{1}{2(1 + \beta_i)}. \tag{10}
\]

According to the above equation, aggregate demand in period 1 is a decreasing function of the interest rate.
Aggregate supply

- Now let’s derive the solution for aggregate supply:

\[ Y_1^s = A_1 L_1^f = A_1 \sum_{i} l_{1,i}^* = A_1 l_{1,i}^* \]  

(11)

\[ = A_1 \sum_{i}^{N} \left[ \frac{1 + 2\beta_i}{2(1 + \beta_i)} - \frac{1}{2(1 + \beta_i)} \frac{1}{1 + R A_1} \right] \]

- In this case we find a positive relationship between the level of aggregate supply and the interest rate. The reason is that agents choose to supply more labor in period 1 if the interest rate rises for some reason.
Equilibrium in the product market

The intersection of the aggregate demand and supply schedules uniquely determines the equilibrium interest rate. Notice that $1 + R$ reflects the relative price of future consumption in terms of current consumption!
Labor supply

- Remember that individual’s labor supply in period 1 is given by:

\[ l_{1,i}^* = \frac{1 + 2\beta_i}{2(1 + \beta_i)} - \frac{1}{2(1 + \beta_i)} \frac{1}{1 + R \frac{w_2}{w_1}}. \]

- Summing over all agents we obtain the expression for aggregate labor supply

\[ L_1^s = \sum_i l_{1,i}^* = \sum_i \left[ \frac{1 + 2\beta_i}{2(1 + \beta_i)} - \frac{1}{2(1 + \beta_i)} \frac{1}{1 + R \frac{w_2}{w_1}} \right] \]

- Not surprisingly aggregate labor supply in period 1 is an increasing function of the wage rate \( w_1 \).
Equilibrium in the labor market

Given that the maximization problem of the firm uniquely determines the wage rate \( w_1 = A_1 \), labor demand is perfectly elastic.
A numerical example

- Consider an economy with 100 agents, so that \( N = 100 \).
- Suppose that 20 of these agents have a discount factor \( \beta_1 = 0.8 \) while the rest of the agents have a discount factor equal to \( \beta_2 = 0.9 \). Notice that the type-2 agents are more patient than the type-1 agents.
- Suppose for simplicity that \( A_1 = A_2 = 1 \).
- Hence in equilibrium \( w_1 = 1 \) and \( w_2 = 1 \) and so \( w_2/w_1 = 1 \).
In any equilibrium aggregate saving is equal to zero and so:

\[ N_1 \frac{\beta_1 A_1 - \frac{A_2}{1+R}}{1 + \beta_1} + N_2 \frac{\beta_2 A_1 - \frac{A_2}{1+R}}{1 + \beta_2} = 0, \]

which implies that

\[ 20 \frac{0.8 - \frac{1}{1+R}}{1 + 0.8} + 80 \frac{0.9 - \frac{1}{1+R}}{1 + 0.9} = 0, \]

and so

\[ 20 \frac{0.8 - \frac{1}{1+R}}{1.8} = 80 \frac{\frac{1}{1+R} - 0.9}{1.9} \]
Equilibrium in the credit market

\[
\begin{align*}
20 \frac{0.8 - \frac{1}{1+R}}{1.8} &= 80 \frac{1}{1+R} - 0.9 \\
\end{align*}
\]

From the above equation it follows that

\[
\begin{align*}
(0.8)20(1.9) - \frac{1.9(20)}{1+R} &= \frac{80(1.8)}{1+R} - (80)0.9(1.8), \\
\end{align*}
\]

or equivalently

\[
\begin{align*}
30.4 - \frac{38}{1+R} &= \frac{144}{1+R} - 129.6. \\
\end{align*}
\]

Collecting terms we get:

\[
\begin{align*}
30.4 + 129.6 &= \frac{144}{1+R} + \frac{38}{1+R} \\
160 &= \frac{182}{1+R} \implies 1 + R = 1.1375 \implies R = 0.1375, (13.75%).
\end{align*}
\]
Consumption

Given our solutions for $w_1$, $w_2$ and $R$, we can now derive the values of consumption:

$$c^*_{1,1} = \frac{1}{2(1 + \beta_1)} \left[ w_1 + \frac{w_2}{(1 + R)} \right]$$

$$= \frac{1}{2(1 + 0.8)} \left( 1 + \frac{1}{1.1375} \right)$$

$$= 0.522.$$

Also,

$$c^*_{2,1} = \beta_1 (1 + R) \frac{1}{2(1 + \beta_1)} \left[ w_1 + \frac{w_2}{(1 + R)} \right]$$

$$= 0.8(1.1375) \frac{1}{2(1 + 0.8)} \left( 1 + \frac{1}{1.1375} \right)$$

$$= 0.475.$$
Consumption

- For the type-2 agents, consumption in period 1 is given by

\[ c_{1,2}^* = \frac{1}{2(1 + \beta_2)} \left[ w_1 + \frac{w_2}{(1 + R)} \right] \]

\[ = \frac{1}{2(1 + 0.9)} \left( 1 + \frac{1}{1.1375} \right) \]

\[ = 0.494. \]

- Similarly, in period 2 these agents consume a quantity:

\[ c_{2,2}^* = \beta_2(1 + R) \frac{1}{2(1 + \beta_2)} \left[ w_1 + \frac{w_2}{(1 + R)} \right] \]

\[ = 0.9(1.1375) \frac{1}{2(1 + 0.9)} \left( 1 + \frac{1}{1.1375} \right) \]

\[ = 0.506. \]
Consumption

- Note that the relatively impatient agents of type 1 consume more in the first period than the patient agents of type 2.
- Summarizing, aggregate demand in the first period is equal to:

\[ C_1^d = 20c^*_1 + 80c^*_2 \]
\[ = 20(0.52198) + 80(0.49451) \]
\[ = 50 \]
Labor supply

- Again let’s start with the agents of type 1:

$$l_{1,1}^* = \frac{1 + 2\beta_1}{2(1 + \beta_1)} - \frac{1}{2(1 + \beta_1)} \frac{1}{1 + RA_1}$$

$$= \frac{1 + 2(0.8)}{2(1 + 0.8)} - \frac{1}{2(1 + 0.8)} \frac{1}{1.1375} = 0.47802$$

- and

$$l_{2,1}^* = \frac{2 + \beta_1}{2(1 + \beta_1)} - \frac{\beta_1(1 + R)A_1}{2(1 + \beta_1)A_2}$$

$$= \frac{2 + 0.8}{2(1 + 0.8)} - \frac{0.8(1.1375)}{2(1 + 0.8)}$$

$$= 0.525.$$
Similarly, for the agents of type 2 we find the following values:

\[ l_{1,2}^* = \frac{1 + 2\beta_2}{2(1 + \beta_2)} - \frac{1}{2(1 + \beta_2)} \frac{1}{1 + R} A_2 \]

\[ = \frac{1 + 2(0.9)}{2(1 + 0.9)} - \frac{1}{2(1 + 0.9)} \frac{1}{1.1375} \]

\[ = 0.50549 \]

and

\[ l_{2,1}^* = \frac{2 + \beta_2}{2(1 + \beta_2)} - \frac{\beta_2(1 + R)}{2(1 + \beta_1)} A_1 \]

\[ = \frac{2 + 0.9}{2(1 + 0.9)} - \frac{0.9(1.1375)}{2(1 + 0.9)} \]

\[ = 0.4935. \]
The labor market

Hence, labor market equilibrium in both periods requires that:

\[ L_1^f = L_1^s \]
\[ = (20l_{1,1}^* + 80l_{1,2}^*) \]
\[ = 20(0.47802) + 80(0.50549) \]
\[ = 50 \]

and

\[ L_2^f = L_2^s \]
\[ = (20l_{2,1}^* + 80l_{2,2}^*) \]
\[ = 20(0.525) + 80(0.4935) \]
\[ = 50 \]
Aggregate demand and supply

en el segundo período. Al agente le importa menos la desutilidad de trabajar que sufre en el futuro. The aggregate supply of goods is given by

\[ Y_1^s = A_1 \left( 20l_{1,1}^* + 80l_{1,2}^* \right) \]
\[ = 20(0.47802) + 80(0.50549) \]
\[ = 50. \]

which is equal to aggregate consumption (demand)

\[ C_1^d = 20c_{1,1}^* + 80c_{1,2}^* \]
\[ = 20(0.52198) + 80(0.49451) \]
\[ = 50 \]
Similarly, in the second period aggregate demand and supply are given by

\[ C^d_2 = 20c^*_{2,1} + 80c^*_{2,2} \]
\[ = 20(0.475) + 80(0.50625) \]
\[ = 50 \]

and

\[ Y^s_2 = A_2 \left( 20l^*_{2,1} + 80l^*_{2,2} \right) \]
\[ = 20(0.525) + 80(0.4935) \]
\[ = 50. \]

Hence, the goods market is in equilibrium in both periods.