Consumption and Savings Decisions: A Two-Period Setting

Dynamic Macroeconomic Analysis

Universidad Autónoma de Madrid

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In the first lecture we studied labor-leisure decisions in a static environment.

In this lecture we study consumption and savings decisions in a two-period setting. The original model was developed by Irving Fisher.

We consider a pure exchange economy. Agents receive a known endowment of goods in both periods.

- Alternatively, we can think of an economy with inelastic labor supply.

The final good is perishable, but agents can borrow and lend on a perfectly competitive credit market.

There is still no money in the economy. No financial intermediation by banks either. Bonds are the only financial instrument.
Savings motives

Why do people save?

1. Consumption smoothing - agents dislike fluctuations in consumption spending.
2. Precautionary motives — fear of unemployment etc.
3. Retirement
4. Purchase of real estate or durable consumption goods.

For the moment we focus on 1. Saving for retirement is treated at the end of the course, while precautionary savings is left for later.
A bond is a piece of paper with a promise of a future payment of a certain amount of goods to the holder of the bond.

Bonds are issued by borrowers and handed over to lenders.

All bonds are supposed to expire in one period.

We denote the (real) interest rate by $R$.

A loan of 1 unit of the final good in period $t$ implies a repayment of $1 + R$ units of the final good in period $t + 1$.

We assume complete information. All loans are repaid.
Notation

- Let $b_t$ denote the bond holdings of a household in period $t$
  - Positive values $b_t > 0$ imply that the household is a lender.
  - On the contrary, when $b_t < 0$ the household is a borrower.
- The household that buys $b_t$ bonds in $t$ will receive $(1 + R_t)b_t$ in period $t + 1$.
- Since the credit market only operates between the first and second period we drop the time index on $R$. 
Budget constraints

- In general terms, the one-period budget constraint of a household can be written as
  \[ c_t + b_t = y_t + (1 + R)b_{t-1} \]
- When \( b_{t-1} > 0 \), the resources of the household exceed the value of income
  \[ y_t + (1 + R)b_{t-1} > y_t \]
- By contrast, when \( b_{t-1} < 0 \), the household has to devote part of this period’s resources to pay back the loan plus interest
  \[ y_t + (1 + R)b_{t-1} < y_t \]
Individual vs. Aggregate Saving

Before we analyze the savings and consumption decisions of Robinson Crusoe, one important remark about the value of aggregate savings.

- With credit markets the consumption of an individual household no longer needs to coincide with her income \( (c_t \neq y_t) \).
- Nonetheless, in any equilibrium of a closed economy aggregate savings must satisfy \( B_t = \sum b_t = 0 \). Why is this the case?
- Consequently, aggregating the individual budget constraints \( c_t + b_t = y_t + (1 + R)b_{t-1} \) we obtain:

\[
C_t = Y_t
\]

- Some households spend more than their income, and others less. But on the aggregate level consumption equals income.
In the rest of this theme, we analyze the case of an economy that lasts for two periods.

- We start by analyzing the decisions of a single agent:
  - Robinson receives an endowment (or fixed income) of $y_1$ units in the first and $y_2$ units in the second period.
  - Robinson takes the interest rate as given and needs to choose his consumption and savings in both periods.
Robinson faces the following pair of one-period budget constraints:

\[
c_1 + b_1 = y_1, \\
\]

\[
c_2 + b_2 = y_2 + (1 + R)b_1.
\]

Robinson’s savings in the second period are necessarily equal to zero — He would like to borrow an infinite amount, but there are no lenders! Hence,

\[
c_2 = y_2 + (1 + R)b_1
\]
The inter-temporal budget constraint

The two budget constraints can be consolidated into a single lifetime budget constraint.

In a first step, we solve for $b_1$:

$$b_1 = y_1 - c_1,$$

Inserting this expression into the right-hand side of the budget constraint for period 2, we obtain

$$c_2 = y_2 + (1 + R)(y_1 - c_1).$$

Dividing both sides by $(1 + R)$ yields the following expression:

$$\left(\frac{c_1}{1 + R} + \frac{c_2}{1 + R}\right) = \left(\frac{y_1}{1 + R} + \frac{y_2}{1 + R}\right).$$

Present Value of Lifetime Consumption = Present Value of Lifetime Income.
Intertemporal budget constraint

- In the sequel we denote the present value of lifetime income (or total resources) by $x$:
  $$x = y_1 + \frac{y_2}{1 + R}$$

- The agent can freely distribute the consumption of these resources over the two periods
  $$c_1 + \frac{c_2}{1 + R} = x.$$  

- Note that $\frac{1}{1 + R}$ is the opportunity cost (in terms of period 1 consumption) of one unit of consumption in period 2.

- Similarly, $(1 + R)x$ is the future value of the agent’s resources and
  $$(1 + R)c_1 + c_2 = (1 + R)x = (1 + R)y_1 + y_2$$
The inter-temporal budget constraint

\[ c_1 + \frac{c_2}{1+R} = x \]

Consumption and Savings

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Preferences

The lifetime budget constraint defines the choice set ("budget set") of the agent. To complete the description of his problem, we need to define his preferences over consumption in both periods.

- We’ll use the following standard specification of time-separable preferences:

\[ U(c_1, c_2) = u(c_1) + \beta u(c_2), \]

- The same function \( u(.) \) defines utility in both periods, but the utility from future consumption is discounted at rate \( \beta \leq 1 \).

- The discount factor is commonly expressed as \( \beta = \frac{1}{1+\rho} \), where \( \rho \) represents the rate of time preference.

- Finally, \( u'(.) = \partial u(.)/\partial c_t > 0 \) and \( u''(.) = \partial u'(.)/\partial c_t < 0 \).
Inter-temporal marginal rate of rate of substitution (IMRS)

The indifference curves in this two-period setting are pairs of current and future consumption that offer the same level of utility $u^0$:

$$U(c_1, c_2) = u(c_1) + \beta u(c_2) = u^0$$

Along the indifference curve

$$u'(c_1) dc_1 + \beta u'(c_2) dc_2 = 0$$

And the IMRS is defined as:

$$\frac{dc_2}{dc_1} = -\frac{u'(c_1)}{\beta u'(c_2)}$$
Example

Throughout the course we will often use the example of logarithmic preferences ("log-utility"): 

\[ u(c_1, c_2) = \log(c_1) + \beta \log(c_2). \]

These preferences satisfy our set of assumptions and often yield simple close-form solutions.

\[ \frac{\partial u(c_1, c_2)}{\partial c_1} = \frac{1}{c_1} ; \quad \frac{\partial^2 u(c_1, c_2)}{\partial c_1^2} = -\frac{1}{c_1^2} \]
Exercise 1

a. Derive the expression for the IRMS in the case of logarithmic preferences:

\[ U(c_1, c_2) = \log(c_1) + \beta \log(c_2) \]

b. How does the value of the IMRS change if we raise the value of \( c_1 \)? What is the explanation?

c. Calculate the value of the IRMS for the case in which \( c_1 = c_2 \).

d. Repeat c for the general case \( U(c_1, c_2) = u(c_1) + \beta u(c_2) \).
Answers

a. The total derivative is given by

\[
\frac{1}{c_1} dc_1 + \frac{\beta}{c_2} dc_2 = 0.
\]

Hence,

\[
\frac{dc_2}{dc_1} = - \frac{c_2}{\beta c_1}
\]

b. An increase in \( c_1 \) causes a fall in the value of the IMRS. Marginal utility from consumption in the first period is decreasing in \( c_1 \). Hence, an agent who consumes a large quantity of goods in the first period and few goods in the second period, will be willing to sacrifice one unit of first-period consumption in return for a small amount of additional consumption in the second period.

c. \( 1/\beta \)

d. Idem
Indifference curves

Consumo mañana

$C_2$

$u_2$

$u_1$

Consumo
The Optimization Problem of an Agent

- In compact form the optimization problem can be written as

\[
\max_{c_1, c_2} [u(c_1) + \beta u(c_2)]
\]  \hspace{1cm} (1)

subject to

\[
\frac{c_2}{1 + R} + c_1 = x.
\]  \hspace{1cm} (2)

- Like before there are two solution methods. Here we use the substitution method. Given that

\[
c_2 = (x - c_1)(1 + R),
\]

we can rewrite the maximand as

\[
\max_{c_1} [u(c_1) + \beta u((x - c_1)(1 + R))]
\]  \hspace{1cm} (3)
The solution

- The FOC associated with our optimization problem

\[
\max_{c_1} \left[ u(c_1) + \beta u((x - c_1)(1 + R)) \right]
\]

is given by:

\[
u'(c_1) + \beta u'((x - c_1)(1 + R))(-1)(1 + R) = 0.
\]

- Or alternatively,

\[
\frac{u'(c_1)}{\beta u'(c_2)} = 1 + R.
\]
Graphical representation of solution

Consumo mañana

\[ x(1+R) \]

\[ c_1 + \frac{c_2}{1+R} = x \]

\[ c_2^* \]

\[ c_1^* \]

\[ x \]
Consumption Euler equation

The optimality condition — known as the Consumption Euler equation —

\[
\frac{u'(c_1)}{\beta u'(c_2)} = (1 + R)
\]

is nothing else than the intertemporal variant of a well-known result in consumption theory.

**IMRS = price ratio of consumption in both periods**
Optimal profile of consumption

According to the Euler equation

\[ u'(c_1) = \beta (1 + R) u'(c_2) \]

there are two opposing forces that affect the inter-temporal choices of the agent

- The stronger the degree of time preference — the lower is \( \beta \) —, the less attractive is \( c_2 \);
- The higher is the interest rate, the more attractive it is to save.
Perfect consumption smoothing

When $\beta(1 + R) = 1$, the Euler eqn. reduces to:

$$u'(c_1) = u'(c_2).$$

Given the strict concavity of $u(.)$, this implies $c_1 = c_2$.

**Intuition**: Recall that $\beta = 1/(1 + \rho)$. Hence, the necessary condition for perfect consumption smoothing can be written as:

$$\beta(1 + R) = \frac{1 + R}{1 + \rho} = 1,$$

which is only satisfied if

$$R = \rho.$$
Rising of falling consumption profiles

Using the same line of argument, one can easily demonstrate that the consumption Euler eqn

\[ \frac{u'(c_1)}{\beta u'(c_2)} = (1 + R) \]

implies the following three results

\[ \beta(1 + R) < 1 \iff c_1 > c_2 \]
\[ \beta(1 + R) = 1 \iff c_1 = c_2 \]
\[ \beta(1 + R) > 1 \iff c_1 < c_2 \]

The above conditions play a central role in any dynamic macro model with endogenous consumption choices.
Log utility

Suppose that the agent’s preferences satisfy

\[ u(c_1) + \beta u(c_2) = \log(c_1) + \beta \log(c_2). \]

In this case, we can write the maximand as

\[ \max_{c_1} [\log(c_1) + \beta \log((x - c_1)(1 + R))] \]

and the associated FOC is:

\[
\frac{1}{c_1} + \beta \frac{1}{c_2} (-1)(1 + R) = 0
\]

\[
\frac{1}{c_1} = \frac{1}{c_2} \beta(1 + R) \implies c_2 = \beta(1 + R)c_1
\]

\[
\frac{c_2}{c_1} = \beta(1 + R).
\]
Log Utility

In this particular case, it is straightforward to obtain closed-form solutions for the optimal consumption levels. Substituting \( c_2 = \beta(1 + R)c_1 \) into the lifetime budget constraint we get:

\[
    c_1 + \frac{c_1 \beta (1 + R)}{1 + R} = x,
\]

And so,

\[
    (1 + \beta)c_1 = x \implies c_1 = \frac{x}{1 + \beta}.
\]

\[
    c_2 = c_1 \beta(1 + R) = \frac{\beta}{1 + \beta}x(1 + R).
\]

\[
    b_1 = y_1 - c_1 = y_1 - \frac{x}{1 + \beta} = \frac{1}{1 + \beta} \left[ \beta y_1 - \frac{y_2}{1 + R} \right]
\]
Lagrange method

- Let’s return to our original problem

\[ \max_{c_1, c_2} \left[ u(c_1) + \beta u(c_2) \right] \]

subject to

\[ \frac{c_2}{1 + R} + c_1 = x. \]

- The Lagrangian associated with the above maximization problem can be written as:

\[ L(c_1, c_2, \lambda) = u(c_1) + \beta u(c_2) + \lambda \left[ x - \frac{c_2}{1 + R} - c_1 \right] \]
Solution

\[ L(c_1, c_2, \lambda) = u(c_1) + \beta u(c_2) + \lambda \left[ x - \frac{c_2}{1 + R} - c_1 \right] \]

The FOCs are given by:

\[ \frac{\partial L(c_1, c_2, \lambda)}{\partial c_1} = u'(c_1) - \lambda = 0 \]
\[ \frac{\partial L(c_1, c_2, \lambda)}{\partial c_2} = \beta u'(c_2) + \lambda \left( -\frac{1}{1 + R} \right) = 0 \]
\[ \frac{\partial L(c_1, c_2, \lambda)}{\partial \lambda} = x - \frac{c_2}{1 + R} - c_1 = 0 \]

Using the fact that \( \lambda = u'(c_1) \), we obtain the same solution as before:

\[ u'(c_1) = u'(c_2) \beta \frac{1 + R}{1 + R} \]

\[ \underbrace{u'(c_1)}_{MC} = \underbrace{u'(c_2) \beta(1 + R)}_{MB} \]
Comparative statics

Our next objective is to determine the response of the optimal solutions to changes in $x$ and $R$.

Let’s start with the case of an increase in $x$:

- A rise in $x$ represents a pure income effect that leads to an increase in current and future consumption;
- The effect on borrowing or lending depends on the timing of the rise in income;
- Transitory and permanent changes produce different effects;
Income effects

Consumo mañana

\[ c_2 \]

\[ c_2' \]

\[ c_2 \]

\[ u_1 \]

\[ u_2 \]

\[ c_1 \]

\[ c_1' \]

\[ x \]

Consumo
Example log utility

Suppose the agent’s preferences are given by:

\[ U(c_1, c_2) = \log(c_1) + \beta \log(c_2) \]

Previously, we have shown that the optimal consumption choices satisfy:

\[ c_1^* = \frac{x}{1 + \beta} \text{ and } c_2^* = \frac{\beta x(1 + R)}{1 + \beta} \]

Hence,

\[ \frac{\partial c_1^*}{\partial x} = \frac{1}{1 + \beta} \Rightarrow 0 \text{ and } \frac{\partial c_2^*}{\partial x} = \frac{\beta(1 + R)}{1 + \beta} \Rightarrow 0 \]
The borrowing/lending position

The changes in \((c_1^*, c_2^*)\) depend solely on the size of \(\Delta x\). By contrast, the changes in \(b_1^*\) depend critically on the timing of the income changes.

- To be more specific, with logarithmic utility,

\[
b_1^* = \beta y_1 - \frac{y_2}{1 + R}.
\]

and so

\[
\frac{\partial b_1}{\partial y_1} = \frac{\beta}{1 + \beta} \Rightarrow 0 \ y \ \frac{\partial b_1}{\partial y_2} = -\frac{1}{(1 + \beta)(1 + R)} = < 0.
\]

- After an increase in \(y_1\) the agent chooses a higher \(b_1^*\). That is, lenders will lend more, while borrowers will borrow less. Similarly, after an increase in \(y_2\) the agents reduce the value of \(b_1^*\).
Interest rate changes

An increase in the interest rate provokes both an income and a substitution effect:

- **substitution effect:** The higher interest rate lowers the cost of consumption in the second period and makes saving more attractive.

- **Income effect:** For given values of $b_1$, the rise in $R$ reduces the feasible consumption levels for borrowers while it raises the wealth of lenders. The former need to pay higher interest payments, while the latter receive higher interest payments.
Higher interest rates

First-period borrowers

\[ D_1 = \text{dotación inicial} \]

\[
x(1+R) \quad x(1+R')
\]

\[
\begin{align*}
\frac{c_1}{1+R} + \frac{c_2}{1+R} &= x \\
&= y_1 y_2
\end{align*}
\]

\[ b_1 < 0 \]
Higher interest rates
First-period lenders

\[ D_2 = \text{dotación inicial} \]

\[ c_1 + \frac{c_2}{1 + R} = x \]

\[ b_1 > 0 \]
The sum of income and substitution effects

The combined effect of a higher interest rate on the consumption levels depends on the sign of $b_1$.

**First-period lenders** ($b_1^* > 0$)

- *Income effect*: $\Delta c_1^* > 0$ & $\Delta c_2^* > 0$
- *Subst. effect*: $\Delta c_1^* < 0$ & $\Delta c_2^* > 0$
- *Combined effect*: $\Delta c_1^* \geq 0$ & $\Delta c_2^* > 0$
- $\Delta c_1^* > 0$ if the income effect dominates the substitution effect.

**First-period borrowers** ($b_1^* < 0$):  

- *Income effect*: $\Delta c_1^* < 0$ & $\Delta c_2^* < 0$
- *Subst. effect*: $\Delta c_1^* < 0$ & $\Delta c_2^* > 0$
- *Combined effect*: $\Delta c_1^* < 0$ & $\Delta c_2^* \geq 0$
- $\Delta c_2^* < 0$ if income effect dominates the substitution effect.
Disentangling income and substitution effects
First-period borrowers

$D_1 = \text{dotación inicial}$

A es la elección original

C es la nueva elección

Desde A hasta B, Efecto Sustitución

Desde B hasta C, Efecto Renta
Disentangling income and substitution effects

First-period lenders

$D_1 = \text{dotación inicial}$

$A$ es la elección original

$E$ es la nueva elección

Desde $A$ hasta $B$, Efecto Sustitución

Desde $B$ hasta $E$, Efecto Renta
Example log utility

With logarithmic preferences, the savings rate does NOT depend on the interest rate, because the income and substitution effect cancel out against each other.

- Recall that

\[ c_1^* = \frac{x}{1 + \beta} \quad \text{and} \quad c_2^* = \frac{\beta x (1 + R)}{1 + \beta}. \]

with \( x = y_1 + \frac{y_2}{1+R}. \)

- Hence, the savings rate \( \frac{c_1^*}{x} = \frac{1}{1+\beta} \) does not depend on \( R. \) But,

\[
\frac{\partial c_1^*}{\partial R} = -\frac{1}{1+\beta} \frac{y_2}{(1 + R)^2} < 0
\]

\[
\frac{\partial c_2^*}{\partial R} = \frac{\beta}{1 + \beta} y_1 > 0
\]
Exercise II

Agents A and B have identical preferences over current and future consumption:

\[ U_i = \ln(c_{1,i}) + \ln(c_{2,i}) \text{ con } i = A, B \]

but they have different endowment patterns as \((y_{1,A}, y_{2,A}) = (1, 0)\) y \((y_{1,B}, y_{2,B}) = (0, 1)\), respectively.

a. Derive the solution for \(c_{1,i}^*\) and \(c_{2,i}^*\) and calculate the exact values of these variables for the case in which \(R = 0\).

b. Analyze the effects of an increase in \(R\).

c. Who benefits from the higher interest rate?
The elasticity of intertemporal substitution

Economists measure the willingness of agents to substitute future for current consumption with the elasticity of intertemporal substitution. The EIS is given by:

\[ EIS = \frac{\partial \ln \left( \frac{c_{t+1}}{c_t} \right)}{\partial \ln \left( \frac{\beta u'(c_{t+1})}{u'(c_t)} \right)} \]

**Computation**

1. Derive the expression for the Intertemporal Marginal Rate of Substitution (IMRS).
2. Calculate the elasticity of the IMRS with respect to the ratio \((c_{t+1}/c_t)\).
3. The EIS is the inverse.
The elasticity of intertemporal substitution

Economists measure the willingness of agents to substitute future for current consumption with the elasticity of intertemporal substitution.

\[
EIS = \frac{\partial \ln \left( \frac{c_{t+1}}{c_t} \right)}{\partial \ln \left( \frac{\beta u'(c_{t+1})}{u'(c_t)} \right)}
\]

Computation

1. Derive the expression for the Intertemporal Marginal Rate of Substitution (IMRS)
2. Calculate the elasticity of the IMRS with respect to the ratio \( \left( \frac{c_{t+1}}{c_t} \right) \).
3. The EIS is the inverse.

The EIS is important because it measures the elasticity of the optimal consumption path to changes in the interest rate.
Example: Log Utility

For the case of logarithmic preferences \( U = \log(c_1) + \beta \log(c_2) \) we have:

\[
IMRS = \frac{dc_2}{dc_1} = -\frac{c_2}{\beta c_1}
\]

The elasticity of the IMRS w.r.t. \( c_2/c_1 \) is given by

\[
\partial \left( -\frac{c_2}{\beta c_1} \right) \frac{c_2}{c_1} = -\frac{1}{\beta} \cdot (-\beta) = 1
\]

The EIS is the inverse of this number and so is 1.
Interest-rate sensitivity of consumption growth

The EIS is of interest because it measures the elasticity of the optimal consumption growth with respect to the interest rate. Given that $IMRS = (1 + R)$ we have:

$$\frac{\left( \frac{d}{c_2^*} \frac{c_2^*}{c_1^*} \right)}{\left( \frac{c_2^*}{c_1^*} \right)} = \frac{d}{d(1 + R)} \cdot \left( \frac{1 + R}{c_2^*} \right)$$

The above statistic can be matched to the observed elasticity in data on consumption.
Log utility

With logarithmic preferences, the growth rate of consumption reduces to:

\[ \left( \frac{c_2^*}{c_1^*} \right) = \frac{\beta x (1+R)}{1 + \frac{x}{1+\beta}} = \beta (1 + R) \]

Hence, the elasticity of intertemporal substitution is equal to

\[ \frac{\partial \beta (1 + R)}{\partial (1 + R)} \cdot \frac{1 + R}{\beta (1 + R)} = 1 \]
Due to concave utility, the agents prefer relatively smooth consumption levels.

Borrowing and lending allows the agents to achieve smooth consumption levels. Indeed, when $\beta(1 + R) = 1$, the agents choose a constant consumption level.

Equilibrium consumption levels depend on the value of $x$. Compensating changes in $y_1$ and $y_2$ that leave $x$ unchanged lead to changes in savings, but they do not alter the optimal consumption levels.
Credit constraints

According to most empirical studies, consumption is more sensitive to fluctuations in current income than suggested by Fisher’s model (“Excess sensitivity of consumption”).

One candidate explanation is the existence of credit constraints. The most extreme form of credit constraints analyzed in the literature assumes

$$b_t \geq 0 \ \forall t$$

Binding credit constraints force agents to lower their consumption in periods with relatively low income, leading to a positive correlation between consumption and current income.
Exercise III

Consider an agent with standard time-separable preferences 
\( U(c_1, c_2) = u(c_1) + \beta u(c_2) \). The agent has a fixed income of 0.5 units of 
the final good in the first period of life and 1.5 units in the second period. 
Suppose for simplicity that \( \beta = 1 \) and \( R = 0 \).

a. Calculate the agent’s optimal consumption decisions in the absence of 
credit constraints.

b. Derive the agent’s consumption choice when agents cannot borrow, 
i.e. \( b_1 \geq 0 \).

c. Repeat a for an agent with an income of 1.5 units in the first and 0.5 
units in the second period.

c. Return to a but now suppose that the agent lives during many 
periods. How could the agent avoid having to reduce consumption in 
bad periods?
N periods / Uncertainty

In real life, agents live for many agents and future income is uncertain. In these circumstances, agents need to solve

$$\max \{c_t, b_t\} \mathbb{E}_0 \sum_{t=0}^{N} \beta^t u(c_t)$$

subject to:

$$c_t + b_t = y_t + (1 + R_t)b_{t-1}$$

Let $\lambda_t$ denote the Lagrange multiplier associated with the period-$t$ budget constraint. Case I: Without uncertainty and with $R_t = R$, the FOCs:

$$\beta^t u'(c_t) = \lambda_t$$

$$0 = -\lambda_t + (1 + R)\lambda_{t+1}$$

$$u'(c_t) = \beta(1 + R_t)u'(c_{t+1})$$
Generalizing lifetime budget constraints

Consider the following sequence of budget constraints

\[\begin{align*}
c_0 + b_0 &= y_0 \\
c_1 + b_1 &= y_1 + (1+R)b_0 \\
&\quad \vdots \\
c_N + b_N &= y_N + (1+R)b_{N-1}
\end{align*}\]

Solving forwards, we obtain:

\[b_N = \sum_{t=0}^{N} \frac{y_t}{(1+R)^t} - \sum_{t=0}^{N} \frac{c_t}{(1+R)^t}\]

When life ends in \(N\), \(b_N = 0\) and we obtain the standard expression.
Uncertain 2-period income

To illustrate the effects of uncertainty, let’s consider a case in which $N = 2$ and income in the second period is uncertain. With $p_l = 0.5$ the agent receives low income $y_{2l}$ and with complementary probability $p_h = 1 - p_l = 0.5$ he receives income $y_{2h} > y_{2l}$. Formally,

$$\max_{c_1, b_1} u(c_1) + \beta \left[ 0.5 u(c_{2l}) + 0.5 u(c_{2h}) \right]$$

s.t. $c_1 + b_1 = y_1$

$$c_{2i} = y_{2i} + (1 + R) b_1$$
Uncertain 2-period income

To illustrate the effects of uncertainty, let’s consider a case in which \( N = 2 \) and income in the second period is uncertain. With \( p_l = 0.5 \) the agent receives low income \( y_{2l} \) and with complementary probability \( p_h = 1 - p_l = 0.5 \) he receives income \( y_{2h} > y_{2l} \). Formally,

\[
\max_{c_1, b_1} u(c_1) + \beta \left[ 0.5 u(c_{2l}) + 0.5 u(c_{2h}) \right]
\]

s.t. \( c_1 + b_1 = y_1 \)

\( c_{2i} = y_{2i} + (1 + R) b_1 \)

F.O.C:

\[
u'(c_1) = \beta (1 + R) \left[ 0.5 u'(c_{2l}) + 0.5 u'(c_{2h}) \right] = \beta (1 + R) E_1 u'(y_{2i} + (1 + R) b_1) < \beta (1 + R) u'(E(y_{2i} + (1 + R) b_1)) \]
Uncertain 2-period income

To illustrate the effects of uncertainty, let’s consider a case in which $N = 2$ and income in the second period is uncertain. With $p_l = 0.5$ the agent receives low income $y_{2l}$ and with complementary probability $p_h = 1 - p_l = 0.5$ he receives income $y_{2h} > y_{2l}$. Formally,

$$\max_{c_1, b_1} u(c_1 + \beta [0.5u(c_{2l}) + 0.5u(c_{2h})])$$

s.t.  $c_1 + b_1 = y_1$

$c_{2i} = y_{2i} + (1 + R)b_1$
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Consumption Euler equation:

$$E_1 \left( \frac{u'(c_1)}{\beta u'(c_{2i})} \right) = 1 + R$$
Nobel prize winner Franco Modigliani considered an extension of the basic model of Fisher to analyze decisions along the life cycle. The basic insight: income varies in an almost deterministic manner along the life cycle and agents use the credit market to insulate consumption from these movements in income.

Agents typically borrow when they are young (education, housing), save during prime age and dissave during retirement.

At different stages of their life, agents or households therefore act at different sides of the credit market.
The Life Cycle Hypothesis

Ingreso y consumo

Perfiles de ingresos a lo largo de la vida

Edad
The Life Cycle Hypothesis

- When the agents have unrestricted access to credit markets, consumption decisions no longer depend on income in a given period but on the PV of lifetime income.
- Let $\bar{y}$ denote average income along the lifecycle and suppose $\beta = 1$ and $(1 + R) = 1$.
- Savings tend to be negative in periods in which $y_t < \bar{y}$.
- Savings tend to be positive in periods in which $y_t > \bar{y}$.
- These optimal savings decisions allow agents to maintain a constant (or smooth) consumption level although income changes.

**Implication:**

- Consumption depends primarily on $\bar{y}$ and hence $x$. The marginal propensity of consumption from changes in $\bar{y}$ is near to 1. By contrast, transitory changes in $y_t$ have little effect on consumption.
Permanent income hypothesis

The life-cycle model of Modigliani focuses on deterministic changes along
the lifecycle. However, during one’s lifetime income also suffers transitory
and largely unpredictable changes in income.

Friedman formalizes this idea by supposing that income in any given
period is the sum of a permanent and a transitory component

\[ y_t = \bar{y} + \eta_t \]

Suppose \( \eta_t \) is i.i.d. with \( E\eta_t = 0 \). In that case it’s clear that \( c_t \) should
hardly respond to the realizations of \( \eta_t \). By contrast, changes in \( \bar{y} \) should
lead to changes in \( c_t \) of similar size.
Bob Hall reconsidered the results of the permanent income hypothesis (PIH) assuming that agents have rational expectations — agents make efficient use of all the available information.

**Main prediction:** If the PIH holds and agents have rational expectations then consumption changes should be unpredictable. That is, consumption follows a random walk and $\mathbb{E}_t c_{t+1} = c_t$.

**In ordinary words:** Agents only revise their consumption decisions if they receive new information that forces them to revise their expectations. This is intrinsically unpredictable.
Ejercicio III

Lucas vive durante dos periodos y su renta disponible en ambos periodos es \( y_1 - \tau_1 \) y \( y_2 - \tau_2 \), donde \( \tau_t \) es un impuesto de cuantía fija en el período \( t \). Suponga que Lucas tiene las siguientes preferencias:

\[
U(c_1, c_2) = \log(c_1) + \log(c_2)
\]

Suponga que el gobierno reduce el valor de \( \tau_1 \) en 1 unidad pero al mismo momento aumenta el impuesto en el segundo período con \( (1 + R) \) unidades.

a. Analice cómo afectan los cambios en los impuestos al valor actual de los recursos totales de Lucas.

b. ¿Cómo afectan los cambios a las decisiones de consumo y ahorro de Lucas?
Equilibrium with Many Agents

To end this second theme, we now proceed with an analysis of equilibrium consumption and savings decisions in an economy with many agents.

- Here we analyze the case of heterogenous agents with different degrees of impatience;
- In the exercises we’ll consider the alternative case in which agents have identical preferences but different income sequences.

In the latter case we are able to generate competitive equilibria in which all agents end up with a constant consumption level during their lifetime.
Setup

- Suppose the economy is populated by a large number of agents. All agents have a constant income stream \( y_1 = y_2 = y \). Hence,

\[
x = y + \frac{y}{1 + R}.
\]

- There are two types of agents with different preferences. \( N_i \) agents of type \( i \) with preferences

\[
\log(c_1) + \beta_i \log(c_2),
\]

and \( N_j \) agents of type \( j \) with preferences

\[
\log(c_1) + \beta_j \log(c_2),
\]

where \( \beta_i < \beta_j \). In other words, the type-\( i \) agents are less patient than their type-\( j \) counterparts.
Individual problem

Recall that the individual optimization problem of type-\(i\) agents can be written as:

\[
\max_{c^i_1} [u(c^i_1) + \beta_i u((x - c^i_1)(1 + R))]
\]

Similarly, the Consumption Euler eqn. is:

\[
c^i_2 = c^i_1 \beta_i (1 + R).
\]

Solutions:

\[
c^i_1 = \frac{x}{1 + \beta_i}
\]

\[
c^i_2 = \frac{\beta(1 + R)x}{1 + \beta_i}
\]

\[
b^i_1 = y - c^i_1 = y - \frac{x}{1 + \beta_i}
\]

\[
= \frac{\beta_i y - \frac{y}{1+R}}{1 + \beta_i}
\]
Credit Market Equilibrium

In any equilibrium, the interest rate adjusts to ensure that the credit market clears. Formally,

\[ N_i b_1^i + N_j b_1^j = 0 \]

Or equivalently,

\[ \underbrace{-N_i b_1^i} \quad = \quad \underbrace{N_j b_1^j} \]

Credit demand \quad Credit supply

Substituting the solutions for \( b_1^i \) and \( b_1^j \) in the market clearing condition yields:

\[ N_i \frac{\beta_i y - \frac{y}{1+R}}{1 + \beta_i} + N_j \frac{\beta_j y - \frac{y}{1+R}}{1 + \beta_j} = 0 \]

Given the values for \((y, \beta_i, \beta_j, N_i, N_j)\) this equation can be solved for \( R \).
A Numerical Example

Suppose that $y = 1$, $\beta_i = 0.8$ and $\beta_j = 0.9$, $N_i = 20$, and $N_j = 80$.

Accordingly, the market clearing condition can be written as:

$$N_i \frac{\beta_i y - \frac{y}{1+R}}{1 + \beta_i} + N_j \frac{\beta_j y - \frac{y}{1+R}}{1 + \beta_j} = 0,$$

$$20 \frac{0.8 - \frac{1}{1+R}}{1 + 0.8} + 80 \frac{0.9 - \frac{1}{1+R}}{1 + 0.9} = 0,$$

So,

$$20 \frac{0.8 - \frac{1}{1+R}}{1.8} = 80 \frac{1}{1+R} - 0.9$$
Solution

\[
20 \frac{0.8 - \frac{1}{1+R}}{1.8} = 80 \frac{1}{1+R} - 0.9
\]

\[
(0.8)20(1.9) - \frac{1.9(20)}{1 + R} = \frac{80(1.8)}{1+R} - (80)0.9(1.8),
\]

\[
30.4 - \frac{38}{1+R} = \frac{144}{1+R} - 129.6.
\]

\[
30.4 + 129.6 = \frac{144}{1+R} + \frac{38}{1+R}
\]

\[
160 = \frac{182}{1 + R} \implies 1 + R = \frac{182}{160} \implies 1 + R = 1.1375
\]
Given \( R \), it is straightforward to calculate the solutions for \( b_1^i \) and \( b_1^j \):

\[
b_1^i = \frac{\beta_i y - \frac{y}{1+R}}{1 + \beta_i} = \frac{0.8 - \frac{1}{1+R}}{1.8} = \frac{0.8 - \frac{160}{182}}{1.8} = \frac{0.8 - 0.879}{1.8} = -0.0439.
\]

\[
b_1^j = \frac{\beta_j y - \frac{y}{1+R}}{1 + \beta_j} = \frac{0.9 - \frac{1}{1+R}}{1.9} = \frac{0.9 - \frac{160}{182}}{1.9} = \frac{0.9 - 0.879}{1.9} = 0.011.
\]

Note also that

\[
N_i b_1^i + N_j b_1^j = (20)(-0.0439) + (80)(0.011) = 0.
\]
Equilibrium Consumption and Savings Decisions

- Besides

\[ x = y + \frac{y}{1 + R} = 1 + \frac{1}{1 + 0.1375} = 1.8791. \]

- Hence,

\[ c_i^i = \frac{x}{1 + \beta_i} = \frac{1.8791}{1.8} = 1.0439 > y_1 = 1. \]

- Obviously, it must be true that

\[ c_i^i + b_i^i = y_1 \\
1.0439 - 0.0439 = 1. \]

- Finally,

\[ c_2^i = \frac{\beta_i x (1 + R)}{1 + \beta_i} \]

\[ = \frac{0.8(1.8791)(1 + 0.1375)}{1 + 0.8} = 0.94999 < y_2 = 1. \]
Equilibrium consumption and savings decisions

- Equivalently, for type-$j$ agents:

\[ c_j^1 = \frac{x}{1 + \beta_j} = \frac{1.8791}{1.9} = 0.989 < y_1 = 1. \]

\[ c_j^1 + b_j^1 = y_1, \]
\[ 0.989 + 0.011 = 1 \]

\[ c_j^2 = \frac{\beta_j x (1 + R)}{1 + \beta_j} \]
\[ = \frac{0.9(1.8791)(1 + 0.1375)}{1 + 0.9} = 1.0125 > y_2 = 1. \]
Summary

- In this second part we have analyzed the competitive equilibrium in an economy of two periods;
- All agents maximize their lifetime utility and can borrow or lend the desired amount at the equilibrium interest rate;
- All individuals enjoy a constant level of income. Nonetheless, in equilibrium
- The impatient agents (type $i$) choose to borrow against their future income;
- The impatient agents (type $j$) choose to lend part of their first-period resources;
- In the exercises you are invited to solve an example in which agents use the credit market to smooth their consumption;