III: Labor Supply in Two Periods

Dynamic Macroeconomic Analysis

Universidad Autoónoma de Madrid

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   - Hechos Estilizados (EE.UU.)
In the first theme we analyzed labor-leisure decisions in a static framework.

Next, we analyzed consumption and savings decisions in a two-period setting with inelastic labor supply.

The logical next step is to endogenize the labor-leisure decisions in this two-period setting with endogenous saving.

For the moment we will take the salaries, denoted by $w_t$, as given.

Agents will sacrifice leisure in periods with relatively high wages and they use the credit market to smooth consumption.

The endogenous determination of salaries is explained in Theme 4.
Budget constraints

Denoting the salaries in period 1 and 2 by $w_1$ and $w_2$, respectively, we can write the one-period budget constraints as:

\[
c_1 + b_1 = w_1 l_1, \\
c_2 = w_2 l_2 + (1 + R) b_1.
\]

The only difference w.r.t. the previous lecture is that income is now endogenous as $y_t = w_t l_t$. The corresponding life-time budget constraint is given by:

\[
c_1 + \frac{c_2}{1 + R} = w_1 l_1 + \frac{w_2 l_2}{1 + R}
\]
Preferences

- To complete the description we need to specify the agent’s preferences over consumption and leisure in the two periods.
- Like before we assume time-separable preferences. In addition we assume that preferences are additive in consumption and leisure. Formally,

\[ U(c_1, l_1, c_2, l_1) = u(c_1) + v(1 - l_1) + \beta[u(c_2) + v(1 - l_2)], \]

where \( c_t \) denotes consumption and \( 1 - l_t \) leisure.
- Note that future consumption and leisure are discounted at the common rate \( \beta \).
- Finally, utility is strictly concave in consumption and leisure.
The optimization problem

- The problem of a representative agent is given by

\[
\max_{c_1, c_2, l_1, l_2} u(c_1) + \nu(1 - l_1) + \beta[u(c_2) + \nu(1 - l_2)]
\]

\[
\text{s.t. } c_1 + \frac{1}{1 + R} c_2 = w_1 l_1 + \frac{1}{1 + R} w_2 l_2
\]

- The Lagrangian associated to the above maximization problem is:

\[
\mathcal{L} = \max_{c_1, c_2, l_1, l_2} \left\{ u(c_1) + \nu(1 - l_1) + \beta[u(c_2) + \nu(1 - l_2)] + \lambda \left[ w_1 l_1 + \frac{1}{1 + R} w_2 l_2 - c_1 - \frac{1}{1 + R} c_2 \right]\right\}, \quad (1)
\]

where \(\lambda\) is the multiplier.
FOCs

There are four FOCs:

\[ c_1 : \quad u'(c_1) - \lambda = 0, \]

\[ c_2 : \quad \beta u'(c_2) - \lambda \frac{1}{1 + R} = 0, \]

\[ l_1 : \quad -v' (1 - l_1) + \lambda w_1 = 0, \]

\[ l_2 : \quad -\beta v' (1 - l_2) + \lambda \frac{w_2}{1 + R} = 0. \]
Intertemporal consumption decisions

Combining the first \((\lambda = u'(c_1))\) and second FOC yields the standard Consumption Euler eqn.:

\[
\frac{u'(c_1)}{MC} = \frac{u'(c_2) \beta(1 + R)}{MB},
\]

Once again, therefore, the agent will choose a constant consumption stream when \(\beta(1 + R) = 1\).
Labor supply

- With the use of the first FOC we can rewrite the FOC associated with $l_1$ in the following manner:

\[
\frac{\partial L}{\partial l_1} \implies -v'(1 - l_1) + \lambda w_1 = 0
\]

\[
v'(1 - l_1) = \underbrace{u'(c_1)w_1}_{MC}
\]

- The above condition equalizes the marginal cost in utility terms from the loss of leisure to the marginal gain from additional consumption.

- Similarly, since $\lambda = u'(c_1)$ and $u'(c_1) = \beta(1 + R)u'(c_2)$:

\[
\frac{\partial L}{\partial l_2} \implies -\beta v'(1 - l_2) + \lambda \frac{w_2}{1 + R} = 0
\]

\[
(\beta) v'(1 - l_2) = \underbrace{(\beta) u'(c_2)w_2}_{MB}
\]
Intertemporal Labor Supply

To obtain the equilibrium levels of labor supply we solve the FOCs for $l_1$ and $l_2$ in terms of $\lambda$:

\[
\frac{\partial L}{\partial l_1} = -v'(1 - l_1) + \lambda w_1 = 0 \implies \\
\lambda = \frac{v'(1 - l_1)}{w_1}
\]

\[
\frac{\partial L}{\partial l_2} = -\beta v'(1 - l_2) + \lambda \frac{w_2}{1 + R} = 0 \implies \\
\lambda = \frac{\beta v'(1 - l_2)(1 + R)}{w_2}.
\]
Equalizing the right-hand sides of both conditions yields

\[
\frac{v'(1 - l_1)}{w_1} = \frac{\beta v'(1 - l_2)(1 + R)}{w_2}
\]

This condition can be rewritten as

\[
\frac{v'(1 - l_1)}{v'(1 - l_2)} = \beta(1 + R)\frac{w_1}{w_2}
\]

The above condition implicitly defines the ratio of labor supply \((l_1 / l_2)\) as an increasing function of the wage ratio \((w_1 / w_2)\).
Logarithmic utility

\[
\max_{c_1, c_2, l_1, l_2} \left\{ \ln c_1 + \ln(1 - l_1) + \beta \left( \ln c_2 + \ln(1 - l_2) \right) + \lambda \left[ w_1 + \frac{1}{1+R} w_2 - c_1 - \frac{1}{1+R} c_2 \right] \right\}.
\]

The associated FOC’s are:

\(c_1\):
\[
\frac{1}{c_1} - \lambda = 0, \quad (2)
\]

\(c_2\):
\[
\beta \frac{1}{c_2} - \lambda \frac{1}{1+R} = 0, \quad (3)
\]

\(l_1\):
\[
-\frac{1}{1-l_1} + \lambda w_1 = 0, \quad (4)
\]

\(l_2\):
\[
-\beta \frac{1}{1-l_2} + \lambda \frac{w_2}{1+R} = 0. \quad (5)
\]
The Optimal Consumption Decisions (given \( x \))

- Combining the first two FOCs (corresponding to \( c_1 \) and \( c_2 \)) we obtain the Consumption Euler eqn. in levels that we have seen before:

\[ c_1 = c_2 \beta (1 + R), \]

- Similarly, like before agents save a fixed proportion \( 1/(1 + \beta) \) of their lifetime resources

\[ x = w_1 l_1 + \frac{w_2 l_2}{1+R} \]

\[ c_1 = \frac{x}{1 + \beta} \]

\[ c_2 = \frac{\beta}{1 + \beta} x (1 + R) \]

- Hence, if we were to take \( x \) as given, the consumption decisions would be the same as before. But \( x \) is NOT exogenous.
Optimal labor supply

Relative labor supply is again a function of the wage ratio $w_1/w_2$, but this time we obtain a condition in levels.

- In a first step we solve the FOCs for $l_1$ and $l_2$ in terms of $\lambda$:

\[
\frac{\partial L}{\partial l_1} \implies -\frac{1}{1-l_1} + \lambda w_1 = 0
\]

\[
\lambda = \frac{1}{w_1(1-l_1)}
\]

\[
\frac{\partial L}{\partial l_2} \implies -\beta \frac{1}{1-l_2} + \lambda \frac{w_2}{1+R} = 0
\]

\[
\lambda = \frac{\beta(1+R)}{w_2(1-l_2)}.
\]
Equalizing the right hand sides of both conditions we obtain the following expression

$$\frac{1}{w_1(1 - l_1)} = \frac{\beta(1 + R)}{w_2(1 - l_2)},$$

which can be rewritten as:

$$\frac{1 - l_1}{1 - l_2} = \frac{1}{(1 + R)\beta w_1} w_2.$$

This arbitrage condition is key for the determination of labor supply in both periods.
Optimal Labor Supply

- First of all, note that

\[ \frac{\partial}{\partial \left( \frac{w_2}{w_1} \right)} \left( \frac{1-l_1}{1-l_2} \right) > 0. \]

- Hence, an increase in the relative wage \( \frac{w_2}{w_1} \) leads to a rise in \( \frac{1-l_1}{1-l_2} \) or equivalently a fall in \( \frac{l_1}{l_2} \).

- The agent decides to consume less leisure in the second period to benefit from the rise in the relative wage. Below we will see that the agent will use the credit market to consume part of the rise in \( w_2/l_2 \) in the first period.

- Formally, the real wage \( w_t \) is a measure of the opportunity cost of leisure in period \( t \). Following an increase in \( w_2/w_1 \) leisure in the second (first) period is therefore relatively more expensive (cheaper).
Optimal Labor Supply

- It is also important to note that
  \[ \frac{\partial}{\partial (1 + R)} \left( \frac{1-t_1}{1-t_2} \right) < 0. \]

- After an increase in the real interest rate the agents demand relatively more leisure in the second period of life.

- First explanation: the higher interest rate reduces the cost of future consumption and raises the return to saving. This is an incentive to work more and save more in the first period of life.

- Second explanation: The rise in \( R \) is equivalent to a fall in the present value of future wages, \( \frac{w_2}{1+R} \). This reduces the opportunity cost of leisure in the second period.
Elasticity of Intertemporal Substitution of Labor

This elasticity measures the percentage change in the ratio of leisure \( \frac{1-l_1}{1-l_2} \) per percent change in the wage ratio \( \frac{w_2}{w_1} \).

\[
\frac{\partial \left[ \left( \frac{1-l_1}{1-l_2} \right) \right]}{\partial \left[ \left( \frac{w_2}{w_1} \right) \right]} = \frac{\partial \left[ \left( \frac{1-l_1}{1-l_2} \right) \right]}{\partial \left[ \left( \frac{w_2}{w_1} \right) \right]} \left( \frac{w_2}{w_1} \right),
\]

When this elasticity is high, agents are willing to substitute a relatively large amount of current leisure for future leisure in response to a rise in the wage ratio \( w_2/w_1 \).
Example: log utility

Previously, we have seen that

$$\frac{1 - l_1}{1 - l_2} = \frac{1}{(1 + R)\beta} \frac{w_2}{w_1}. $$

Hence,

$$\frac{\partial}{\partial \left( \frac{1 - l_1}{1 - l_2} \right)} \left( \frac{w_2}{w_1} \right) = \frac{1}{(1 + R)\beta}. $$

and so,

$$\frac{\partial}{\partial \left( \frac{1 - l_1}{1 - l_2} \right)} \left( \frac{w_2}{w_1} \right) \frac{1 - l_1}{1 - l_2} = \frac{1}{(1 + R)\beta} \frac{w_2}{w_1} \frac{1}{(1 + R)\beta} = 1. $$

In other words, the elasticity is equal to 1 and so a $x$ percent increase in $(\frac{w_2}{w_1})$ leads to a $x\%$ rise in $(\frac{1 - l_1}{1 - l_2})$. 
Optimal consumption levels

- Combining the FOCs for $l_1$ and $c_1$

\[-\frac{1}{1 - l_1} + \lambda w_1 = 0.\]

\[\frac{1}{c_1} = \lambda,\]

- we obtain the following equality

\[\frac{1}{1 - l_1} = \frac{1}{c_1} w_1.\]

- Hence,

\[c_1 = w_1 - w_1 l_1 \implies w_1 l_1 = w_1 - c_1.\]
Optimal Consumption Levels

- Similarly, combining the FOC for $l_2$

$$-\beta \frac{1}{1 - l_2} + \lambda \frac{w_2}{1 + R} = 0.$$  

and the condition that $\lambda = \frac{1}{c_1}$, we obtain:

$$\beta \frac{1}{1 - l_2} = \frac{1}{c_1} \frac{w_2}{1 + R}.$$  

- Hence,

$$c_1 \beta (1 + R) = w_2 - w_2 l_2,$$

and so:

$$w_2 l_2 = w_2 - c_1 \beta (1 + R) \implies \frac{w_2 l_2}{1 + R} = \frac{w_2}{1 + R} - c_1 \beta.$$
Optimal consumption levels

- Since

\[ w_1 l_1 = w_1 - c_1 \]
\[ \frac{w_2 l_2}{1 + R} = \frac{w_2}{1 + R} - c_1 \beta. \]

we get

\[ x = w_1 l_1 + \frac{w_2 l_2}{1 + R} \]
\[ = w_1 - c_1 + \frac{w_2}{1 + R} - c_1 \beta. \]

- Using the fact that \( c_1 = \frac{x}{1+\beta} \), we can therefore write:

\[ (1 + \beta) c_1 = w_1 - c_1 + \frac{w_2}{1 + R} - c_1 \beta, \]
Solutions for $c_1^*$ and $c_2^*$

The final solutions:

$$c_1^* = \frac{1}{2(1 + \beta)} \left[ w_1 + \frac{w_2}{(1 + R)} \right].$$

$$c_2^* = c_1 \beta (1 + R) = \beta (1 + R) \frac{1}{2(1 + \beta)} \left[ w_1 + \frac{w_2}{(1 + R)} \right].$$
Solution for $l_1^*$

- To obtain the value for $l_1^*$, we substitute the expression for $c_1^*$ into the condition $w_1 l_1 = w_1 - c_1$,

\[
w_1 l_1 = w_1 - c_1 = w_1 - \frac{1}{2(1 + \beta)} \left[ w_1 + \frac{w_2}{(1 + R)} \right]
\]

- Hence,

\[
l_1^* = \frac{1 + 2\beta}{2(1 + \beta)} - \frac{1}{2(1 + \beta)} \frac{1}{1 + R} \frac{w_2}{w_1}.
\]

Comparative statics: $\partial l_1^*/\partial w_1 > 0$; $\partial l_1^*/\partial w_2 < 0$; $\partial l_1^*/\partial R > 0$
Solution for $l_2^*$

- Similarly, since $wl_2 = w - c_1 \beta (1 + R)$,

$$w_2l_2 = w_2 - c_1 \beta (1 + R)$$

$$= w_2 - \frac{\beta (1 + R)}{2(1 + \beta)} \left[ w_1 + \frac{w_2}{(1 + R)} \right]$$

$$= w_2 \left[ 1 - \frac{\beta (1 + R)}{2(1 + \beta)} \frac{1}{1 + R} \right] - \frac{\beta (1 + R)}{2(1 + \beta)} w_1.$$

- So,

$$l_2^* = \frac{2 + \beta}{2(1 + \beta)} - \frac{\beta (1 + R)}{2(1 + \beta)} \frac{w_1}{w_2}.$$

Comparative statics: $\frac{\partial l_2^*}{\partial w_1} < 0$ ; $\frac{\partial l_2^*}{\partial w_2} > 0$ ; $\frac{\partial l_2^*}{\partial R} < 0$
A special case

Suppose for the moment that $\beta = 1$ and $R = 0$.

- The constant consumption level satisfies:

$$c_1^* = \frac{1}{2(1 + \beta)} \left[ w_1 + \frac{w_2}{1 + R} \right] = \frac{1}{4} [w_1 + w_2] = c_2^*,$$

while the solutions for $l_1^*$ and $l_2^*$ reduce to:

$$l_1^* = \frac{1 + 2\beta}{2(1 + \beta)} - \frac{1}{2(1 + \beta)} \frac{1}{1 + R} \frac{w_2}{w_1}$$

$$= \frac{3}{4} - \frac{1}{4} \frac{w_2}{w_1}$$

$$l_2^* = \frac{2 + \beta}{2(1 + \beta)} - \frac{\beta(1 + R)}{2(1 + \beta)} \frac{w_1}{w_2}$$

$$= \frac{3}{4} - \frac{1}{4} \frac{w_1}{w_2}.$$
A special case

- Given the optimal choices for $l_t$, total labor income satisfies

$$x = w_1 l_1 + w_2 l_2$$

$$= w_1 \left( \frac{3}{4} - \frac{1}{4} \frac{w_2}{w_1} \right) + w_2 \left( \frac{3}{4} - \frac{1}{4} \frac{w_1}{w_2} \right)$$

$$= w_1 \frac{3}{4} - \frac{1}{4} w_2 + w_2 \frac{3}{4} - \frac{1}{4} w_1$$

$$= \frac{1}{2} (w_1 + w_2)$$

and the agent consumes half of $x$ in each period.

- Finally, if $w_2 = w_1$, then

$$l_1 = l_2 = \frac{1}{2}.$$

Hence, the agent works half time in both periods and consumes his or her entire labor income, so $b_1^* = 0$. 
Concluding remarks

- In this section we have analyzed labor supply and consumption decisions in a two-period setting with exogenous prices.
- For given labor supply choices, the consumption decisions are the same as before, but labor supply is endogenous and we need to solve the jointly optimal consumption and labor supply decisions.
- The optimizing agents will concentrate their labor supply in periods with a relatively high wage rate and they will use the credit market to smooth consumption over time.
- The elasticity of intertemporal substitution of labor measures the responsiveness of labor-leisure choices to changes in the relative wages $w_2/w_1$.
- Suppose, realistically, that wages are high in booms and low in recessions. This would lead to pro-cyclical fluctuations in labor supply.
Salario Medio por Hora (USA 2000)
Horas Trabajadas y Salario Medio por Hora (USA 2000)

Graphs by Sex

Male
Female

Hourly wage
Weekly hours worked

Age
Hourly Wage
Weekly hours worked
Horas Trabajadas y Salario Medio por Hora (Mujeres-USA 2000)

![Graphs showing Hourly Wage and Weekly Hours Worked for Never Married and Ever Married individuals.](image)

Graphs by evermar
Salario horario por Nivel de Educación (USA 2000)

Dynamic Macroeconomic Analysis (UAM)
III: Labor Supply in Two Periods
Horas Trabajadas por Nivel de Educación (USA 2000)

La gráfica muestra el número medio de horas trabajadas por nivel de educación en Estados Unidos en el año 2000. Se observa una tendencia general de aumento en el número de horas trabajadas con la edad, aunque hay diferencias notables entre los niveles educativos.

- **Less HS**: Menos de secundaria
- **HS**: Secundaria
- **Some college**: Alguna educación universitaria
- **College +**: Educación universitaria más allá de asociado

La curva para aquellos con menos de secundaria muestra un aumento inicial rápido que se estabiliza alrededor de la edad de 25 años. La curva para aquellos que completaron secundaria muestra un aumento más gradual que también se estabiliza alrededor de la edad de 25 años. Las curvas para aquellos con educación universitaria son aún más ligeras y se estabilizan a una edad más tardía, siendo el nivel de educación universitaria más allá de asociado el que muestra la curva que se estabiliza más tarde y a un nivel más alto de horas trabajadas.
Salario y Horas Trabajadas por Nivel de Educación (USA 2000)

[Graphs showing mean hourly wage and weekly hours of work by education level (Less HS, HS, Some College, College+)]

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Horas Trabajadas y Salario Medio (Histórico Anual)
Tendencias en Horas Trabajadas
Cambio Porcentual 1970-2002

Source: OECD Employment Outlook, 2004