Equilibrium in a Model with Overlapping Generations with Physical Capital

Dynamic Macroeconomic Analysis

Universidad Autónoma de Madrid

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1. OLG with physical capital

2. The basic OLG model
   - Consumption and savings decisions
   - Firms
   - The household problem
   - Un Ejemplo

3. Equilibrium
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4. Fluctuaciones
   - Temporary TFP shocks
OLG with physical capital

- In the previous lecture we analyzed a basic OLG model with perishable goods.

- In this lecture we assume that the final good can be used as capital.

- Household work and save when young. At old age, the household offer their savings to firms (capital) and at the end of their life they consume their entire income.

- Again we will see that the equilibrium is generically inefficient.

- The setup will be used to explain the (i) the persistence of productivity shocks (ii) the long-run effects of taxation (iii) the (dis)advantages of a PAYGO pension scheme.
Specific objectives

- Derivation of the steady state equilibrium for our benchmark OLG model with physical capital
  - In the absence of shocks, the economy converges to a steady state and remains there forever.
- The propagation and persistence of TFP shocks in a model with physical capital.
  - Temporary shocks to TFP have persistent effects because they affect capital accumulation.
  - The model replicates some stylized facts about the co-movements of consumption, investment and output at business cycle frequencies.
- Topics: taxation, dynamic (in)efficiency and PAYGO vs. capitalized pension schemes.
The basic setup

- We will consider an economy in which agents (households) live for two periods.
- The agents of generation $t$ are young in $t$ and old in $t+1$.
  - When young, they coincide with the old agents of generation $t-1$.
  - When old, they live with the young generation of $t+1$.
- There is a unique good that can either be consumed or saved as capital.
- Given that the agents only live for two periods, all saving is done by the young generation.
- For the moment each dying household is replaced by one young household. At any moment in time there are $N$ young households, and $N$ old households.
Labor supply and savings

For the moment we assume that labor supply is inelastic. Moreover, the agents only work during the first period of their lives.

- Each young agent offers one unit of labor to a representative firm.
  - Labor supply is completely inelastic. The young agents only need to decide on their savings and consumption.
  - Aggregate savings of the young serve as the stock of capital in the next period.

- The old do not work (retirement).
  - They rent their savings (capital) to the representative firm and consume all their resources before they die.
The Firm

- In each period there is a representative firm that hires capital from the old and labor services from the young to produce the unique final good. The firm is endowed with a neo-classical production function \( Y_t = A_t F(K_t, L_t) \).

- The firm pays a competitive wage to the young and a rental price (interest) to the owners of capital. Thus, profits are equal to

\[
\Pi = A_t F(K_t, L_t) - w_t L_t - r_t K_t
\]

- Suppose the firm (farm) produces wheat. In each period it combines the seeds of the old and the manual labor of the young to produce wheat. Part of the harvest is paid to the young in the form of salaries. The rest of the harvest is paid to the old (principal plus interest) who consume all their income.
Before we proceed, it is important to highlight the assumptions of physical capital.

- Capital is owned by households and rented to firms in a competitive market. This assumption is standard in dynamic macromodels.

- Capital depreciates at rate $\delta$ — in each period a percentage $\delta$ of the capital stock “melts” away.

- Gross investment is equal to the savings of the young. Net-investment is gross investment minus depreciation.
Consumption and savings decisions

The intertemporal budget constraint

- Let’s consider the problem of a representative agent of generation $t$. The agent offers one unit of labor and receives a wage of $w_t$. Hence, the income of the young is simply $w_t$.

- The agent needs to decide how much he wants to consume this period, $c_{yt}$, and how much he wants to save for his old age, $s_{t+1}$. Note that the savings of generation $t$ carry a subscript $t + 1$. These are goods saved for the future.

- Hence, the first-period budget constraint can be written as:

\[
\begin{align*}
    w_t &= c_{yt} + s_{t+1} \\
    s_{t+1} &= w_t - c_{yt}
\end{align*}
\]
The inter-temporal budget constraint

• As mentioned before, the old agents of any generation consume all their resources:

\[ c_{ot+1} = (1 - \delta) s_{t+1} + s_{t+1}r_{t+1} \]
\[ = s_{t+1} (1 + r_{t+1} - \delta) \]

• The old receive the amount of capital rented to firms \(s_{t+1}\) plus interest \(s_{t+1}r_{t+1}\). Moreover, capital depreciates at rate \(\delta \in (0, 1)\) per period. Hence \((1 - \delta) s_{t+1}\) is the value of capital net of depreciation.

• The old consume the net value of their savings \((1 - \delta) s_{t+1}\) plus interest \(s_{t+1}r_{t+1}\). Total consumption is \(s_{t+1}(1 + r_{t+1} - \delta)\)
The inter-temporal budget constraint

- Combining the budget constraints for the two periods we get

\[ c_{ot+1} = s_{t+1} (1 + r_{t+1} - \delta) \]
\[ = (w_t - c_{yt}) (1 + r_{t+1} - \delta) \]

- Hence, the inter-temporal budget constraint for a member of generation \( t \) is given by

\[ \frac{c_{ot+1}}{(1 + r_{t+1} - \delta)} = (w_t - c_{yt}) \]
\[ c_{yt} + \frac{c_{ot+1}}{(1 + r_{t+1} - \delta)} = w_t \]

- In words, the discounted present value of lifetime consumption is equal to the wage.
The inter-temporal budget constraint

\[ c_{yt} - c_{ot+1} + w_t (1+r_{t+1}-d) = (w_t - c_{yt})(1+r_{t+1}-d) \]

Figure 2a
Preferences

- As usual, we represent the preferences of the agent by means of a utility function:

  \[ U(c_{yt}, c_{ot+1}) = u(c_{yt}) + \beta u(c_{ot+1}), \text{ with } \beta < 1, \; u' > 0, \; u'' < 0, \]

- In most cases we will consider the example of logarithmic preferences

  \[ U(c_{yt}, c_{ot+1}) = \log(c_{yt}) + \beta \log(c_{ot+1}) \]
Preferences

\[ c_{ot+1} \]

\[ c_{yt} \]
Firms

- There is a large number of firms that combine capital and labor to produce the unique final good using a technology that is represented by

\[ Y_t = A_t F(K_t, L_t), \]

- The neo-classical production function \( F \) is strictly increasing and concave in both arguments:

\[
\frac{\partial F_t}{\partial K_t} > 0, \quad \frac{\partial F_t}{\partial L_t} > 0, \quad \frac{\partial^2 F_t}{\partial K_t^2} < 0, \quad \frac{\partial^2 F_t}{\partial L_t^2} < 0.
\]

\( F \) also exhibits constant returns to scale (CRS), i.e.

\[ F(\lambda K_t, \lambda L_t) = \lambda F(K_t, L_t), \quad \text{for all } \lambda > 0, \]

and satisfies the Inada conditions

\[
\lim_{K \to 0} F_K = \lim_{L \to 0} F_L = \infty, \quad \text{and} \quad \lim_{K \to \infty} F_K = \lim_{L \to \infty} F_L = 0.
\]
The representative firm

- Since the production function exhibits CRS, we can assume **without loss of generality** that there is a single representative firm. The objective of the firm is to maximize its profits

\[
\max_{L_t, K_t} Y(K_t, L_t) - w_t L_t - r_t K_t.
\]

- The FOC’s are given by

\[
w_t = \frac{\partial Y_t}{\partial L_t} = MPL; \quad r_t = \frac{\partial Y_t}{\partial K_t} = MPK
\]

- The representative firm hires capital (labor) until the marginal product of capital (labor) is equal to the rental price of capital (wage).
Profits

- The assumption of a CRS production function guarantee that the equilibrium profits of the representative firm are equal to zero.

- This result is a direct consequence of the so-called Euler theorem which establishes that if $F(K_t, L_t)$ is homogenous of degree 1, then

$$F(K_t, L_t) = \frac{\partial F}{\partial K_t} K_t + \frac{\partial F}{\partial L_t} L_t = r_t K_t + w_t L_t$$

- For example, when $F(K, L) = AK^\alpha L^{1-\alpha}$ we have

$$F_K K + F_L L = [\alpha AK^{\alpha-1} L^{1-\alpha}] K + [(1 - \alpha) AK^\alpha L^{-\alpha}] L = F(K, L)$$
**Timeline of decisions**

<table>
<thead>
<tr>
<th>Período t-1</th>
<th>Período t</th>
<th>Período t+1</th>
</tr>
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<tbody>
<tr>
<td>[jóvenes] generación t-1</td>
<td>[jóvenes] generación t</td>
<td>[jóvenes] generación t</td>
</tr>
<tr>
<td>Reciben salario ($w_{t-1}$), Consumen ($c_{t-1}$), Ahorran ($s_t$)</td>
<td>Reciben salario ($w_t$), Consumen ($c_t$), Ahorran ($s_{t+1}$)</td>
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<tr>
<td>Trabajo</td>
<td>FIRMA</td>
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<tr>
<td>capital</td>
<td>(1+$r_t$-$d$) $s_t$</td>
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<td>[viejos] Proveen capital a la firma ($s_t$)</td>
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<tr>
<td>Consumen el ingreso total((1+$r_t$-$d$) $s_t$)</td>
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</table>
The household problem

- Agents born in $t$ ($\forall t \geq 1$) solve the following problem

$$\max \quad u(c_y, t) + \beta u(c_{o,t+1})$$

s.t. $c_{y,t} + s_{t+1} \leq w_t,$

$$c_{o,t+1} \leq (1 + r_{t+1} - \delta) s_{t+1}. $$

- Combining the two budget constraints, we obtain the familiar problem:

$$\max_{c_{y,t}, c_{o,t+1}} \quad u(c_y, t) + \beta u(c_{o,t+1})$$

s.t. $c_{y,t} + \frac{c_{o,t+1}}{1 + r_{t+1} - \delta} = w_t.$
The household problem

- The Langrangean associated with the household problem is given by

\[ L = \max_{c_{yt}, c_{ot+1}} u(c_{yt}) + \beta u(c_{ot+1}) + \lambda \left[ w_t - c_{yt} - \frac{c_{ot+1}}{1+r_{t+1} - \delta} \right] \]

- The FOCs are given by

\[ u'(c_{yt}) = \lambda \]

\[ \beta u'(c_{ot+1}) = \frac{\lambda}{1 + r_{t+1} - \delta} \]

- The only difference with before is the discount factor which includes the net-return on capital.
Consumption Euler condition

In our OLG setup, the consumption Euler condition of generation-$t$ households is given by:

$$\beta (1 + r_{t+1} - \delta) u'(c_{o,t+1}) = u'(c_{y,t})$$

In principle, this equation should include an expectation sign as households need to decide the value of $s_{t+1}$ before they know the realization of $r_{t+1}$.

- In a deterministic setting one could assume perfect foresight.
- In a stochastic setting, the common assumption is rational expectations.
- Here we avoid these complications — we only consider examples in which $s_{t+1}$ does not depend on $r_{t+1}$. 
Our classroom example

- The economy is populated by $N$ individuals of each generation. Hence, in each period there is a population of size $2N$ ($N$ young and $N$ old).

- The agents have logarithmic preferences

  $$ u(c_{yt}, c_{ot+1}) = \log(c_{yt}) + \beta \log(c_{ot+1}). $$

- The production technology of the representative firm can be represented by a Cobb-Douglas production function:

  $$ Y_t = A_t K_t^\alpha L_t^{1-\alpha}. $$
The firm problem

- Let’s start with the solution of the static firm problem:

\[
\max_{K_t, L_t} A_t K_t^\alpha L_t^{1-\alpha} - w_t L_t - r_t k_t,
\]

- In previous lectures we have seen that the FOC’s are given by:

\[
w_t = (1 - \alpha) A_t K_t^\alpha L_t^{-\alpha} = (1 - \alpha) A_t \left( \frac{K_t}{L_t} \right)^\alpha,
\]

\[
r_t = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} = \alpha A_t \left( \frac{K_t}{L_t} \right)^{\alpha-1}.
\]
The household problem

- Recall that the consumption-savings decision of any generation $t \geq 1$ is given by:

$$
\max_{c_{yt}, c_{ot+1}} \log(c_{yt}) + \beta \log(c_{ot+1})
$$

subject to

$$
c_{yt} + \frac{c_{ot+1}}{1 + r_{t+1} - \delta} = w_t.
$$

- With logarithmic preferences the consumption Euler condition

$$
\begin{align*}
\underbrace{u'(c_{yt})}_{\text{MC}} &= \underbrace{\beta(1 + r_{t+1} - \delta)u'(c_{ot+1})}_{\text{MB}}. \\
\end{align*}
$$

simplifies to

$$
\frac{1}{c_{yt}} = \beta(1 + r_{t+1} - \delta) \frac{1}{c_{ot+1}}.
$$

- Hence,

$$
c_{ot+1} = \beta c_{yt}(1 + r_{t+1} - \delta).
$$
The optimal savings ratio

To show that the equilibrium savings ratio does not depend on the interest rate, we write the Euler condition as

\[ c_{o t+1} = \beta c_{y t} (1 + r_{t+1} - \delta). \]

Substituting the above condition in the budget constraint we get:

\[ c_{y t} + \frac{\beta c_{y t} (1 + r_{t+1} - \delta)}{1 + r_{t+1} - \delta} = c_{y t} + \beta c_{y t} = w_t. \]

Hence,

\[ c_{y t} = \frac{1}{1 + \beta} w_t, \]

\[ s_{t+1} = w_t - \frac{1}{1 + \beta} w_t = \frac{\beta}{1 + \beta} w_t \]
The household problem

The solution

1

\[ c_{yt} = \frac{1}{1 + \beta} w_t, \quad (3) \]

2

\[ c_{ot+1} = \beta c_{yt} (1 + r_{t+1} - \delta) = \frac{\beta(1 + r_{t+1} - \delta)}{1 + \beta} w_t, \quad (4) \]

3

\[ s_{t+1} = w_t - c_{yt} = w_t - \frac{1}{1 + \beta} w_t = \frac{\beta}{1 + \beta} w_t. \quad (5) \]

NB: savings depend on the value of \( w_t \) and NOT on future values of the interest rate!
Dynamics

- Modern macro-models are characterized by very complicated dynamics as the links between periods run in both directions
  - Current decisions about savings and investment affect future opportunities and prices.
  - Beliefs about future prices and returns affect current decisions.

- Here we simplify matters by adopting a specification in which equilibrium savings and investment only depend on current prices.
  - Thus the dynamic interactions run in one direction: today’s decisions define the state of the economy tomorrow.

- Nonetheless, the dynamics are interesting as they mimic the dynamics of the Solow growth model
  - The savings ratio of the young is constant and after an initial period of growth the economy converges to a steady state with zero growth.
  - Short-term (fluctuations) and long-term phenomena (growth) can be analyzed in the same model.
The equilibrium of the economy

The resource constraint

- Note that the total resources in the economy in period $t$ are equal to

  $$A_t K_t^\alpha L_t^{1-\alpha} + (1 - \delta) K_t$$

- In equilibrium these resources are used for the consumption of the old in period $t$, $N_{c_{ot}}$, consumption of the young in the same period $N_{c_{yt}}$, and savings of the young, $N_{s_{t+1}}$.

- Accordingly, the equilibrium condition for the goods market can be written as:

  $$N_{c_{yt}} + N_{c_{ot}} + N_{s_{t+1}} = A_t K_t^\alpha L_t^{1-\alpha} + (1 - \delta) K_t, \forall t \geq 1, \ (6)$$
The equilibrium of the economy

The resource constraint

The market clearing condition for the goods market was shown to be:

\[ N_{cyt} + N_{cot} + N_{st+1} = A_t K_t^\alpha L_t^{1-\alpha} + (1 - \delta) K_t, \forall t \geq 1, \]

Recall that the savings of the young constitute the capital stock of the next period:

\[ N_{st+1} = K_{t+1} \text{ for } t \geq 1. \] (7)

Hence, the market clearing condition can be rewritten as

\[ N_{cyt} + N_{cot} + K_{t+1} = A_t K_t^\alpha L_t^{1-\alpha} + (1 - \delta) K_t, \text{ for } t \geq 1. \] (8)
Definition of equilibrium

We are now in a position to define the equilibrium:

- Given any sequence of parameters \( \{A_t\}_{t=1}^{\infty} \), an equilibrium consists of a sequence of prices \( \{w_t, r_t\}_{t=1}^{\infty} \), and assignments of the households \( \{c_{yt}, c_{ot+1}, s_{t+1}\}_{t=1}^{\infty} \) and firms \( \{K_t, L_t\}_{t=1}^{\infty} \) such that:
  - Given \( \{w_t, r_t\}_{t=1}^{\infty} \), the households of all generations \( t \geq 1 \) maximize their expected utility so that \( \{c_{yt}, c_{ot+1}, s_{t+1}\} \) satisfy conditions (3), (4) and (5).
  - Given \( \{w_t, r_t\}_{t=1}^{\infty} \), the firms maximize their profits in all periods, i.e. \( K_t \) and \( L_t \) satisfy conditions (1) and (1) \( \forall t \geq 1 \).
  - All markets clear.
Market clearing conditions

- Labor market:
  \[ L_t = N. \]

- The goods market:
  \[ Nc_{yt} + Nc_{ot} + Ns_{t+1} = A_t K_t^\alpha L_t^{1-\alpha} + (1 - \delta) K_t, \]

- The capital market:
  \[ Ns_{t-1} = K_t. \]
Labor market equilibrium

- Notice that the equilibrium in the labor market is trivial. The agents do not value leisure and so they are willing to work “full time” for any positive salary and in equilibrium $L_t = N \forall t \geq 1$.
- Furthermore, we know that $w_t$ is simply the MPL in period $t$:

$$w_t = (1 - \alpha) A_t K_t^\alpha L_t^{1-\alpha} = (1 - \alpha) A_t \left(\frac{K_t}{N}\right)^{\alpha}.$$  

which depends positively on the capital-labor ratio $\kappa_t = K_t / N$. 

Equilibrium consumption of the young and old

The solution for the wage rates allows us immediately to calculate the consumption levels of the young (generation $t$) and old (generation $t - 1$) in $t$:

$$c_{yt} = \frac{1}{1 + \beta} w_t = \frac{1}{1 + \beta} (1 - \alpha) A_t \left( \frac{K_t}{N} \right)^\alpha$$

$$c_{ot} = (1 + r_t - \delta) \frac{\beta}{1 + \beta} w_{t-1} = \frac{\beta(1 + r_t - \delta)}{1 + \beta} (1 - \alpha) A_t \left( \frac{K_{t-1}}{N} \right)^\alpha.$$
Goods market equilibrium

- As shown before, the market clearing condition for the goods market is

\[ Nc_{yt} + Nc_{ot} + Ns_{t+1} = A_t K_t^\alpha L_t^{1-\alpha} + (1 - \delta) K_t. \]

- Aggregate consumption in \( t \) is equal to \( C_t = Nc_{yt} + Nc_{ot} \). Similarly, since \( Ns_{t+1} = K_{t+1} \), we can define gross investment as

\[ I_t = K_{t+1} - (1 - \delta) K_t: \]

\[ Nc_{yt} + Nc_{ot} + K_{t+1} - (1 - \delta) K_t = A_t K_t^\alpha L_t^{1-\alpha}. \]
Equilibrium allocations

- Notice that $C_t = Y_t - I_T$ and so

$$C_t = A_t K_t^\alpha L_t^{1-\alpha} - K_{t+1} + (1 - \delta) K_t$$

$$= A_t K_t^\alpha N^{1-\alpha} - N \frac{\beta}{1 + \beta} w_t + (1 - \delta) K_t$$

$$= A_t K_t^\alpha N^{1-\alpha} - N \frac{\beta}{1 + \beta} (1 - \alpha) A_t \left( \frac{K_t}{N} \right)^\alpha + (1 - \delta) K_t$$

$$= \left( 1 - \frac{\beta}{1 + \beta} (1 - \alpha) \right) A_t K_t^\alpha N^{1-\alpha} + (1 - \delta) K_t$$

$$= \left( 1 - \frac{\beta}{1 + \beta} (1 - \alpha) \right) Y_t + (1 - \delta) K_t$$
Similarly, $I_t$ is given by

\[
I_t = Y_t - C_t
\]
\[
= A_t K_t^\alpha N^{1-\alpha} - \left(1 - \frac{\beta}{1 + \beta} (1 - \alpha)\right) A_t K_t^\alpha N^{1-\alpha} - (1 - \delta) K_t
\]
\[
= \frac{\beta}{1 + \beta} (1 - \alpha) A_t K_t^\alpha N^{1-\alpha} - (1 - \delta) K_t.
\]
\[
= \frac{\beta}{1 + \beta} (1 - \alpha) Y_t - (1 - \delta) K_t.
\]
Our objective is to analyze convergence to steady state in an economy in which $A_t = 1$.

In this economy we know that $s_{t+1}$, $w_t$ and $K_{t+1}$ satisfy:

$$s_{t+1} = \frac{\beta}{1 + \beta} w_t,$$

$$w_t = (1 - \alpha) \left( \frac{K_t}{N} \right)^{\alpha},$$

$$K_{t+1} = N \frac{\beta}{1 + \beta} (1 - \alpha) \left( \frac{K_t}{N} \right)^{\alpha} = \frac{\beta}{1 + \beta} (1 - \alpha) N^{1-\alpha} K_t^{\alpha}, \quad (9)$$

Condition (9) defines the Law of Motion for the capital stock.
Steady state equilibrium

- A steady state equilibrium is a situation in which the equilibrium allocation is the same in every period.

- In other words, if the capital stock at the start of a period \( t \) is equal to \( \bar{K} \) then the equilibrium must be such that the young in period \( t \) decide to save exactly a quantity \( Ns_{t+1} = \bar{K} \).

- In the case of our benchmark economy we can demonstrate that the equilibrium converges uniformly to a unique steady state.

- Intuitively, for any economy with \( K_t < \bar{K} \) we find that \( Ns_{t+1} = K_{t+1} > K_t \). Conversely for any \( K_t > \bar{K} \) the young agents will choose a lower capital stock and so \( K_{t+1} < K_t \).
Convergence to steady state

Inspection of the law of motion for capital

\[ K_{t+1} = N \frac{\beta}{1 + \beta} (1 - \alpha) \left( \frac{K_t}{N} \right)^{\alpha} = \frac{\beta}{1 + \beta} (1 - \alpha) N^{1-\alpha} K_t^\alpha, \]

reveals that the right-hand side is a strictly concave function of \( K_t \).

The above equation is called a law of motion because it defines the value of next period’s capital stock as a function of this period’s capital stock.

In fact the pair \((A_t, K_t)\) completely defines the outcome of the economy in this period, and in the absence of shocks it also defines the state of the economy in period \( t + 1 \). Formally, state variables are pre-determined variables.
Convergence

Notice that the law of motion is given by

\[ K_{t+1} = N \frac{\beta}{1 + \beta} (1 - \alpha) \left( \frac{K_t}{N} \right)^\alpha = \frac{\beta}{1 + \beta} (1 - \alpha) N^{1-\alpha} K_t^\alpha, \]

Hence, the slope of this expression is equal to:

\[ \frac{\partial K_{t+1}}{\partial K_t} = \frac{\alpha \beta}{1 + \beta} (1 - \alpha) N^{1-\alpha} K_t^{\alpha-1} \]
\[ = \frac{\alpha}{1 + \beta} (1 - \alpha) \left( \frac{N}{K_t} \right)^{1-\alpha} \]

This is a continuous function of \( K_{t+1} = g(K_t) \) with

\[ \lim_{K_t \to 0} \left( \frac{\partial K_{t+1}}{\partial K_t} \right) = \infty, \lim_{K_t \to \infty} \left( \frac{\partial K_{t+1}}{\partial K_t} \right) = 0 \] and \( g(0) = 0 \). These features guarantee convergence.
Convergence
Calculation of the steady state capital stock

In steady state $K_{t+1} = K_t = K$. Substituting this equality in our law of motion

$$K_{t+1} = N \frac{\beta}{1 + \beta} (1 - \alpha) \left( \frac{K_t}{N} \right)^{\alpha} = \frac{\beta}{1 + \beta} (1 - \alpha) N^{1-\alpha} K_t^\alpha,$$

yields

$$K = \frac{\beta}{1 + \beta} (1 - \alpha) N^{1-\alpha} K^{\alpha}.$$

$$K^{1-\alpha} = \frac{\beta}{1 + \beta} (1 - \alpha) N^{1-\alpha}$$

$$K = \left( \frac{\beta}{1 + \beta} (1 - \alpha) \right)^{\frac{1}{1-\alpha}} N$$

$$\frac{K}{N} = \left( \frac{\beta}{1 + \beta} (1 - \alpha) \right)^{\frac{1}{1-\alpha}}$$
Properties of the steady state

- Economies with the same fundamentals \((\alpha, \beta)\) converge to the same steady state.

- The steady state capital stock is strictly increasing in the constant average savings ratio \(\frac{\beta}{1+\beta}(1-\alpha)\). The same is not true for steady state consumption (of the young):
  - The young obtain a fraction \((1-\alpha)\) of national income which is the labor share.
  - They save a fraction \(\frac{\beta}{1+\beta}\) of their income.

- A permanent increase in the savings rate leads to a temporary growth in output and a permanent increase in the steady state level of \(K\).

- Permanent growth in living standards (per capita consumption) requires a permanent increase in \(A\).
A quick comparison with the Solow growth model

The process of capital accumulation and convergence in our OLG model is similar to the mechanics of the Solow growth model. However, there are relevant differences:

- In the Solow model, there are infinitely-lived agents who save an arbitrary fraction of their income.
- In our model the agents have finite lifetimes and they save an optimal fraction of labor income.
  - The fact that the old consume their capital is isomorphic to a 100% depreciation rate.
  - This assumption places a very strict upper-limit on the capital stock,
A first glance at the response to TFP shocks

To obtain a rough idea about the response of consumption to TFP shocks, we can calculate the elasticity of $C_t$ w.r.t. $A_t$. First notice that:

$$\frac{\partial Y_t}{Y_t} = \frac{\partial Y_t}{\partial A_t} = \frac{1}{K_t L_t^{1-\alpha}} = 1,$$

Next,

$$C_t = \left[1 - \frac{\beta}{1+\beta} (1-\alpha)\right] Y_t + (1-\delta) K_t$$

$$\frac{\partial C_t}{C_t} = \frac{\left(1 - \frac{\beta}{1+\beta} (1-\alpha)\right) K_t A_t N^{1-\alpha}}{\left(1 - \frac{\beta}{1+\beta} (1-\alpha)\right) A_t K_t^\alpha N^{1-\alpha} + (1-\delta) K_t} < 1$$
A first glance at the effects of TFP shocks

- Similarly, in the case of investments we obtain

\[ I_t = \frac{\beta}{1+\beta} (1 - \alpha) Y_t - (1 - \delta) K_t \]

\[ \frac{\partial I_t}{I_t} = \frac{\left[ \frac{\beta}{1+\beta} (1 - \alpha) K_t^\alpha N^{1-\alpha} \right] A_t}{\frac{\beta}{1+\beta} (1 - \alpha) A_t K_t^\alpha N^{1-\alpha} - (1 - \delta) K_t} > 1 \]

- In sum, in response to a TFP shock the ratio between the percentage changes in \( C \) and \( Y \) is smaller than 1, while investments fluctuate relatively more than output.

- The above features mimic regularities of actual business cycle fluctuations.
Temporary TFP shocks

- In the next slides we will analyze the effects of a temporary shock to TFP in an economy that is initially in steady state.

- In the first step we will calculate the deterministic steady state in which \( A_t = 1 \) for all \( t \).

- Then we consider the effects of an unanticipated increase in \( A \) that lasts for one period.

- Our objective is to show that this shock has persistent effects: the economy does not immediately return to steady state.
The baseline

- Suppose that $\beta = 1$, $\alpha = 0.3$, $\delta = 0.1$, and $N = 100$. In this economy, the steady state capital stock is given by:

$$K = \left( \frac{\beta}{1 + \beta} (1 - \alpha) \right) \frac{1}{1 - \alpha} N$$

$$= \left( \frac{1}{2} (1 - 0.3) \right) \frac{1}{1 - 0.3} 100 = 22.319$$

- The steady-state level of output is:

$$Y = K^\alpha L^{1-\alpha} = (22.319)^{0.3} (100)^{1-0.3} = 63.768$$

- Aggregate consumpton in steady state is

$$C = \left( 1 - \frac{\beta}{1 + \beta} (1 - \alpha) \right) K^\alpha N^{1-\alpha} + (1 - \delta) K$$

$$= \left( 1 - \frac{1}{2} (1 - 0.3) \right) (22.319)^{0.3} (100)^{1-0.3} + (1 - 0.1) 22.319$$

$$= 61.536$$
The steady state level of gross investment is:

\[
I = \frac{\beta}{1 + \beta} (1 - \alpha) A_t K_t^\alpha N^{1-\alpha} - (1 - \delta) K_t
\]

\[
= \frac{1}{2} (1 - 0.3) (22.319)^{0.3} (100)^{1-0.3} - (1 - 0.1)22.319
\]

\[
= 2.2319
\]

Notice that in steady state

\[
I = \delta K = (0.1)22.319 = 2.2319,
\]

Investment serves entirely to compensate depreciation.
We now consider the effects of an unanticipated TFP shock. In a given period $t$, the level of $A_t$ jumps up to 1.05.

In $t + 1$, the value of $A$ returns to its initial value of 1 and stays there forever.

The fact that the shock is unanticipated rules out any possible anticipation effects.

So, $K_t = K = 22.319$ and the effect of the TFP shock is a proportional increase in $Y_t$:

$$Y_t = A_t K_t^\alpha N^{1-\alpha}$$

$$= (1.05) (22.319)^{0.3} (100)^{1-0.3} = 66.956$$
The response of consumption

- The higher output leads to an increase in aggregate consumption:

\[
C_t = \left(1 - \frac{\beta}{1 + \beta}(1 - \alpha)\right) A_t K_t^\alpha N^{1-\alpha} + (1 - \delta) K_t
\]

\[
= \left(1 - \frac{1}{2}(1 - 0.3)\right) (1.05) (22.319)^{0.3} (100)^{1-0.3} + (1 - 0.1) 22.319
\]

\[
= 63.609
\]

- Notice that the total amount of resources and hence consumption increase less than proportionally because \(K_t\) is pre-determined.
The response of investment

- The higher level of income also leads to higher investments as the young raise their savings

\[ I_t = \frac{\beta}{1 + \beta} (1 - \alpha) A_t K_t^\alpha N^{1-\alpha} - (1 - \delta) K_t \]

\[= \frac{1}{2} (1 - 0.3) (1.05) (22.319)^{0.3} (100)^{1-0.3} - (1 - 0.1) 22.319 \]

\[= 3.3476 \]

- Notice that \( I \) increase by \( 100 \times (3.3476 - 2.2319)/2.2319 = 50\% \! \)

- Intuition: in steady state \( I = 0.1K \). But due to the TFP shock \( w_t, s_{t+1} \) and \( K_{t+1} \) all increase by 5\%. 

The response of the capital stock

- Next period’s capital stock, $K_{t+1}$, is equal to

$$K_{t+1} = (1 - \delta)K_t + I_t$$
$$= (1 - 0.1)22.319 + 3.3476 = 23.435$$

- Consequently, $Y_{t+1} > Y$ although $A_{t+1} = 1$:

$$Y_{t+1} = A_{t+1}K_{t+1}^\alpha L^{1-\alpha}$$
$$= 1(23.435)^{0.3}(100)^{1-0.3} = 64.708$$

Note que incluso si el shock dura un sólo período, $Y_{t+1}$ es todavía mayor que $Y = 63.768$. 
Persistence

- The fact that $Y_{t+1}$ and $K_{t+1}$ stay above their steady state values, implies that $C_{t+1}$ also stays above the level of $C$:
  
  $$C_{t+1} = \left(1 - \frac{\beta}{1 + \beta} (1 - \alpha) \right) Y_{t+1} + (1 - \delta) K_{t+1}$$

- Can we say the same thing about $I_{t+1}$?
  
  $$I_{t+1} = \frac{\beta}{1 + \beta} (1 - \alpha) Y_{t+1} - (1 - \delta) K_{t+1}.$$  

  That’s not clear. On the one hand, $Y_{t+1}$ is high, and so the young can save a high amount. But on the other hand, $K_{t+1}$ is also high and this reduces the need to invest.
Temporary TFP shocks

- The figure on the next slide shows the impulse responses — i.e. the percentage deviations of $C_t$, $I_t$, $K_t$, $Y_t$ from their steady state values in the periods after the shock.

- For example, the first-period deviation in $Y_t$ is defined as:

$$\left( \frac{Y_1 - Y}{Y} \right) \times 100 = \left( \frac{66.956 - 63.768}{63.768} \right) \times 100 = 4.9\%$$

The rest of the deviations is defined in a similar manner.
Deviations

Percentage deviations from steady state values
Summary

- Consumption is shown to be less responsive to TFP shocks than investment (when measured in perc. deviations from steady state). This is consistent with the data.
- The effects of the shock last for several periods (persistence). This is due to inter-temporal substitution.
- The agents do not want to consume the entire increase in income in period $t$. As a result, savings and investment increase.
- The increase in investment leads to a rise in the capital stock and this (stock) variable returns sluggishly to its steady state value.
Real business cycle theory

- Our OLG-model produces an expansion period after a one-period TFP shock.
- It is easy to show that a negative TFP shock would lead to a recessionary period in which consumption, income and output would remain below their steady state values for several periods.
- Recurrent shocks would lead to fluctuations around steady state.
- Due to the optimal response of the agents the effect of the shocks propagate through the economy and may persist after the shock has disappeared.
- This propagation and amplification of real shocks is the focus of the so-called real business cycle theory. The inter-temporal substitution of labor and consumption places a prime role in these theories.
Modern business cycle theory

Nowadays all modern business cycle models use setups with rational, forward-looking and maximizing agents.

Contributions to the RBC literature assume perfectly flexible prices and wages and analyze the role of real shocks (and frictions).

Modern neo-Keynesian models use the same setup but assume rigid prices and/or wages. The agents take these frictions in price setting into account when they take their decisions.

Both strands of the business cycle literature do have radically different policy implications.
In the standard RBC-model labor supply is endogenous and equilibrium hours of work fall in recessions and rise in booms.

- Can we interpret this drop in hours as unemployment?
- Should the government intervene to avoid the fluctuations in labor supply and consumption?
Welfare

- In an OLG-setting it is far from trivial to talk about welfare.

- We will see examples in which there is no controversy because the simple introduction of an intrinsically worthless commodity ("money") may lead to Pareto improvements.
  - Some generations — current and future unborn generations — can be made strictly better off without making any other generation worse off.

- In most cases, however, we need to make comparisons between utility gains for some generations and utility losses for other generations.

- The prime explanation for the scope of inefficient outcomes: a bewildering number of missing markets.
  - Current generations cannot trade with unborn generations. Governments, on the contrary, can and often should take the interests of future generations into account.
Golden rule capital stock

A first approximation to the issue of welfare is to ask whether the decisions of the agents maximize the steady state level of consumption. In the growth literature, this level is known as the Golden Rule Capital Stock. Recall that the resource constraint in our economy is given by

\[
N\omega t + N\omega_{0t} + (K_{t+1} - (1 - \delta)K_t) = A_t K_t^\alpha N^{1-\alpha}
\]

\[
C_t + I_t = A_t K_t^\alpha N^{1-\alpha}
\]

Moreover, in steady state \(K_{t+1} = K_t = K\) and so \(I_t = I = \delta K\). Accordingly, let \(K^{GR}\) define the Golden Rule capital stock. It solves

\[
\max_K C = AK^\alpha N^{1-\alpha} - \delta K
\]
Golden Rule Capital Stock

The level of $K^{GR}$ is implicitly defined by the condition:

$$\alpha A \left[ \frac{N}{K^{GR}} \right]^{1-\alpha} = \delta$$

$$MPK = \delta$$

Comparing the equilibrium and the golden rule capital stock for $A = 1$, we find:

$$K^{GR} = \left( \frac{\alpha}{\delta} \right)^{\frac{1}{1-\alpha}}$$

$$K^* = \left[ (1 - \alpha) \frac{\beta}{1 - \beta} \right]^{\frac{1}{1-\alpha}} N$$

When $\delta = 1$ both capital stocks coincide iff. $\alpha = \frac{\beta}{1+\beta} (1 - \alpha)$. 
Insufficient savings

- Whenever $K^* < K^{GR}$, steady state consumption is below its maximum level.
- In any given period a government could confiscate the savings of the old and transfer these resources to the young.
- Measures of this kind would have a positive transitory effect on the next generations
- However, this “trick” cannot be repeated — the young would anticipate that their savings will be confiscated and savings would drop to zero.
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In the standard RBC-model labor supply is endogenous and equilibrium hours of work fall in recessions and rise in booms.

- Can we interpret this drop in hours as unemployment?
- Should the government intervene to avoid the fluctuations in labor supply and consumption?
Suppose that the economy of a small island is well described by our OLG model. The agents have logarithmic preferences and only work during the first period of their lives. The representative firm has a Cobb-Douglas production function, $Y_t = A_t K_t^\alpha L_t^{1-\alpha}$ and each generation consists of $N$ agents.

Suppose that the economy finds itself in steady state when it is hit by a *tsunami* that destroys half of the capital stock. All the inhabitants survive and for simplicity we assume that the economy returns to normality after one period (except for the loss of capital) and $A_t = 1 \forall t$. 

Example

1. How does the tsunami affect factor prices, $w_t$ and $r_t$, and output, $Y_t$, in period $t$?

2. Derive the expression for the consumption of young and old agents in period $t$ assuming that the tsunami hits the economy at the start of $t$ before production takes place. How are these consumption levels affected by the tsunami?

3. How does the tsunami affect the savings decisions of the young?

4. The destructions lead to a 50% in the capital stock. What can we say about the percentage drop in output? It is bigger or smaller than the drop in $K_t$.

5. How do you expect the capital stock and output to change in the successive periods? And what are in your opinion the long-run effects of the tsunami? Will the economy fully recover?
In equilibrium, \( w_t \) and \( r_t \) are equal to the marginal product of labor and capital, respectively:

\[
\downarrow w_t = (1 - \alpha) A_t K_t^\alpha L_t^{-\alpha} = (1 - \alpha) A_t \left( \downarrow \frac{K_t}{L_t} \right)^\alpha, (10)
\]

and

\[
\uparrow r_t = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} = \alpha A_t \left( \downarrow \frac{K_t}{L_t} \right)^{\alpha-1}. (11)
\]

Thus, the fall in the capital/labor ratio leads to a fall in the wage and a rise in the rental price of capital.
Solution

Notice that we can actually be quite precise. Since $A_1 = 1 \forall t$:

$$w_t = (1 - \alpha) \left( \frac{0.5K}{N} \right)^{\alpha} = 0.5^\alpha w$$

$$r_t = \alpha \left( \frac{N}{0.5K} \right)^{1-\alpha} = 0.5^\alpha r$$

$$w_t N = 0.5^\alpha wN$$

$$r_t K_t = 0.5^{\alpha-1} r(0.5K) = 0.5^\alpha rK$$

$$Y_t = w_t N + r_t K_t = 0.5^\alpha (wN + rk)) = 0.5^\alpha Y$$

Note also that

$$Y_t = (0.5K)^{\alpha} N^{1-\alpha} = 0.5^\alpha K^\alpha N^{1-\alpha}$$
Consumption

How does the tsunami affect consumption of the young and old?
For the young generation we find:

\[ c_{yt} = \frac{1}{1 + \beta} w_t = 0.5^\alpha c_y \]

\[ s_{t+1} = \frac{\beta}{1 + \beta} w_t = 0.5^\alpha s_t \]

\[ K_{t+1} = Ns_{t+1} = 0.5^\alpha K \geq 0.5K \forall \alpha < 1 \]

The old get a higher return on capital but only half of their capital stock survived:

\[ Nc_{ot} = Nr_t K_t + 0.5(1 - \delta)K = 0.5^\alpha NrK + 0.5(1 - \delta)K \]
Solution

We have already shown that \( K_t < K_{t+1} = 0.5^\alpha K < K \). Consequently,

\[
Y_{t+1} = (0.5^\alpha K)^\alpha N^{1-\alpha}
\]

\[
0.5^{\alpha^2} K^\alpha N^{1-\alpha}
\]

Suppose for example that \( \alpha = 1/3 \). Then,

\[
Y_{t+1} = 0.5^{1/9} K^{1/3} N^{2/3} = 0.9259 Y
\]

\[
K_{t+2} = 0.5^{1/9} K
\]

\[
Y_{t+2} = (0.5^{1/9} K)^{1/3} N^{2/3} = 0.5^{1/27} K^{1/3} N^{2/3} = 0.9747 Y
\]

Eventually, the economy returns to its initial steady state. Both the initial drop and the speed of convergence depend on \( \alpha \).
Questions

- Can you think of a policy intervention in $t$ that would speed up the process of recovery? How would this policy affect the various generations alive in period $t$? Does the same policy produce the same effects if $t$ is repeated in later periods?
- Suppose that labor supply is endogenous. How could this affect the transmission of the shock?
- Suppose there is a nearby island that is unaffected by the shock. Suppose that the two islands are identical. Where should the young of this island invest their savings after the tsunami?
- Suppose on the contrary that half of the young agents on the tsunami-stricken island take a boat and leave the island to work elsewhere. How would this change the recovery process?