1. Consider a version of the Solow growth model (in continuous time) with population growth and no technological progress. The agents in this economy own the capital stock and save a fraction $\sigma \in (0, 1)$ of their income. The production technology is described by

$$Y_t = AK_t^\alpha L_t^{1-\alpha},$$

where $A$ denotes total factor productivity, $K_t$ denotes the capital stock, and $L_t$ denotes both the size of the population and the number of workers. We assume that $\partial L/\partial t = \dot{L}_t = n$, while capital is assumed to depreciate at rate $\delta$.

a. Write down the expression for $\dot{K}_t$ and use this expression to derive the expression for $\dot{\kappa}_t$ where $\kappa_t = K_t/L_t$ is the capital-labour ratio at $t$. (15 points)

b. Derive the expression for the steady-state value of $\kappa$. (15 points)

c. Derive the value of the Golden Rule-savings ratio, $\sigma^{GR}$, that maximizes the steady-state value of per-capita consumption, $c$. Your answer should include the formal derivation of the Golden Rule capital-labour ratio. (10 points)

d. Analyze the implications of an increase in the population growth rate for the steady-state values of $\kappa$ and $c$ when $\sigma = \sigma^{GR}$. Illustrate your answer with a figure. (10 points)
2. Consider a pure exchange economy of two periods. All agents have the following preferences over current and future consumption, $c_1$ and $c_2$ respectively:

$$U(c_1, c_2) = \sqrt{c_1} + \beta \sqrt{c_2},$$

with $0 < \beta \leq 1$. The $N_a$ agents of type $a$ receive a known endowment of 1 unit of the final good in period 1 and 0 in period 2, i.e. $y^a_1 = 1$ and $y^a_2 = 0$. Similarly, the $N_b$ agents of type $b$ receive an endowment stream of $y^b_1 = 0$ and $y^b_2 = 1$. The agents have access to a perfectly competitive credit market and we denote the real interest rate by $R$.

a. Derive the expression for the Inter-temporal Marginal Rate of Substitution $dc_2/dc_1$ (10 points).

b. Solve the optimization problem of a representative agent for arbitrary values of $R$, $\beta$ and the present value of lifetime resources, $x$. Your answer should include the expressions for the optimal consumption levels $c^*_1, c^*_2$ and bond holdings $b^*_1$ (or savings) (15 points).

c. Write down the equilibrium condition for the credit market and derive the equilibrium interest rate $R$ for the case in which $N_b = N_a$ and $\beta = 1$. How much do the agents consume in both periods? (15 points)

d. Identify the necessary condition for an equilibrium in which $\beta(1 + R) = 1$ and all agents choose a constant consumption profile (10 points).