1 a. The agents maximize the utility function subject to the following lifetime budget constraint:

\[ c_{y,t} + \frac{c_{o,t+1}}{1 + R_{t+1} - \delta} = w_t l_t \]

The FOCs are given by

\[ \frac{1}{c_{y,t}} = \lambda \]
\[ \frac{\beta}{c_{o,t+1}} = \frac{\lambda}{1 + R_{t+1} - \delta} \]
\[ \frac{1}{1 - l_t} = \lambda w_t \]

where \( \lambda \) is the Lagrange multiplier.

Combining the first two FOCs we obtain the standard Euler consumption equation:

\[ c_{o,t+1} = \beta(1 + R_{t+1} - \delta)c_{y,t} \]

Reinserting this equation into the budget constraint we obtain the following intermediate solutions:

\[ c_{y,t} = \frac{1}{1 + \beta} w_t l_t \]
\[ c_{o,t+1} = \frac{\beta}{1 + \beta}(1 + R_{t+1} - \delta)w_t l_t \]
\[ s_{t+1} = \frac{\beta}{1 + \beta} w_t l_t \]

Finally, combining the first and the third FOC we obtain

\[ \frac{1}{1 - l_t} = \frac{w_t}{c_t} \]

Inserting our intermediate solution for \( c_{y,t} \) into the above expression we get

\[ l_t = \frac{1 + \beta}{2 + \beta} \]

Finally, from the maximization problem of the firm

\[ \Pi = AK_t^{\alpha} L_t^{1-\alpha} - w_t L_t - R_t K_t \]
we obtain the standard equality between factor prices and marginal product:

\[ w_t = (1 - \alpha)A_t K_t^{\alpha} L_t^{-\alpha} \]
\[ R_t = \alpha A_t K_t^{\alpha-1} L_t^{-\alpha} \]

b. The savings of the young in \( t \) provide the capital stock in \( t + 1 \) and in equilibrium firms voluntarily hire all the capital and labour:

\[ K_{t+1} = N s_{t+1} \]
\[ = N \frac{\beta}{1+\beta} w_t l_t \]
\[ = N \frac{\beta}{1+\beta} (1 - \alpha) K_t^{\alpha} (N l_t)^{-\alpha} l_t \]

Collecting terms and imposing the steady state condition \( K_{t+1} = K_t \) we get

\[ K = \left[ (1 - \alpha) \frac{\beta}{1+\beta} \right]^{\frac{1}{1-\alpha}} N l_t \]
\[ = \left[ (1 - \alpha) \frac{\beta}{1+\beta} \right]^{\frac{1}{1-\alpha}} N \frac{1+\beta}{2+\beta} \]
\[ Y = \left[ (1 - \alpha) \frac{\beta}{1+\beta} \right]^{\frac{1}{1-\alpha}} (N l_t)^{\alpha} (N L_t)^{-1-\alpha} = \left[ (1 - \alpha) \frac{\beta}{1+\beta} \right]^{\frac{1}{1-\alpha}} (N l_t) \]
\[ w = (1 - \alpha) \left[ (1 - \alpha) \frac{\beta}{1+\beta} \right]^{\frac{1}{1-\alpha}} (N l_t)^{\alpha} (N l_t)^{-\alpha} = (1 - \alpha) \left[ (1 - \alpha) \frac{\beta}{1+\beta} \right]^{\frac{1}{1-\alpha}} \]

c. As the shock was unanticipated \( K_T = K \) and since the equilibrium supply of labour is invariant, output drops 5% below its steady state value. Moreover, the same is true for wages and hence savings. The latter means that \( K_{T+1} = 0.95K \) and so

\[ Y_{T+1} = K_T^{\alpha} (N l)^{1-\alpha} = (0.95K)^{\alpha} (N l)^{1-\alpha} = 0.95^{\alpha} K^{\alpha} (N l)^{1-\alpha} \]
\[ K_{t+2} = 0.9^\alpha K \]
\[ Y_{T+2} = (0.95^\alpha K)^{\alpha} (NL)^{1-\alpha} = 0.95^{\alpha^2} Y \]

d. In the case of a logarithmic utility function the income and substitution effects from a change in the wage rate cancel out against each other, so that \( l_t \) is constant over time and independent of productivity shocks. By contrast, for utility specifications in which the substitution effect dominates the income effect, the agents will work less (more) in periods in which \( A_t \) falls below (rises above) its steady state value and these pro-cyclical movements in labour amplify the fluctuations in output.

2 This problem illustrates clearly that the OLG structure creates missing markets. The young would like to lend part of their first-period endowments to agents who could repay them in the future, but the problem is there are no such agents. The current old will not be around next period, the future young are not born yet and none of the current young are willing to borrow as they all face the same problem.
a. Formally, the Planner’s Problem can be written as

\[
\max \sum_{t=0}^{\infty} [u(c_{y,t}) + u(c_{o,t+1})]
\]
\[
s.t \quad Nc_{y,t} + NC_{o,t+1} = N\omega
\]

Notice that the resource constraint can be rewritten as \( c_{y,t} + c_{o,t} = \omega \), i.e. the endowment of one young agent needs to be shared between the young agent and one old agent. Notice also that there are no prices - the planner can freely distribute the available resources.

The FOC for \( c_{y,t} \) and \( c_{o,t+1} \) are given by

\[
u'(c_{y,t}) = \lambda_t \]
\[
u'(c_{o,t+1}) = \lambda_{t+1}
\]

Equivalently, for generation \( t + 1 \) we find

\[
u'(c_{y,t+1}) = \lambda_{t+1}
\]
\[
u'(c_{o,t+2}) = \lambda_{t+2}
\]

Since all generations receive the same treatment, it must hold that \( c_{y,t} = c_{y,t+1} \) and so \( \lambda_t = \lambda_{t+1} \) and this in turn implies

\[
u'(c_{y,t}) = \nu'(c_{o,t+1})
\]

and so \( c_{y,t} = c_{o,t+1} = 0.5 \).

b. As explained in the heading, the young agents would like to save part of their first-period endowment and lend it to another agent in return for repayment in the next period. But due to the OLG structure there are no borrowers in the economy that will be alive tomorrow. As a result, in the competitive equilibrium, all young agents consume their endowment when young and they have zero consumption when old.

c. If the government introduces a PAYGO pension system, they can impose a social security contribution of \( \tau = 0.5\omega \) on the \( N \) young agents of any generation \( t \) and pay a pension \( b = 0.5\omega \) to the old of generation \( t - 1 \). Next, in period \( t + 1 \) the old of generation \( t \) receive a pension payment of \( b = 0.5\omega \). Formally, the two budget constraints of the agents can be written as

\[
c_{y,t} = \omega - \tau
\]
\[
c_{o,t+1} = b
\]

and the PAYGO pension system is self-financing as \( N\tau = Nb = 0.5N \).