Networks of strings: modelization and control of vibrations

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Abstract

We present some results related to the observability and controllability of systems of vibrating strings, which appear in connection with the study of observability and controllability properties of networks of strings.

I. INTRODUCTION

A network of strings is a system where several elastic strings are coupled at their ends. The mathematical model of the motion of a network is constituted by several $1 - d$ wave equations describing the motion of every string and some coupling boundary conditions at the points where the strings are coupled.

These networks, are a particular case, and perhaps the simplest one, of the so-called multi-structures: a structure formed by several vibrating elements as strings, beams, membranes, plates. The corresponding model is a system of coupled Partial Differential Equations.

In last years, special attention has been paid to the controllability problem for multi-structures, that is, to the possibility of driving to rest the vibrations of the structure acting on a small part of it. A quite complete introduction to this topic is the book [LLS2], where a large number of models of multi-structures can also be found. We have provided a rather wide bibliography where the interested reader can find some of the most important developments, which have been made in this subject.

A particular type of control problem for multi-structures with practical relevance is the simultaneous control, when the same controlling action is applied to different elements, not necessarily coupled through other conditions. In these notes we shall address mainly two simultaneous control problems for a system constituted by two elastic strings: the simultaneous boundary control and simultaneous internal control. Though these problem are very similar, there is an important difference in the controllability properties of the corresponding systems. When two strings are simultaneously controlled from one end, it is never possible to drive to rest all the initial deformations of the strings. This may be done only for the elements of some subspace that depends on the ratio between the lengths of the strings. On the other hand, when the simultaneous control is applied on an interior region of the strings, then all the initial states can be driven to rest. The control possibilities of the interior mechanism turn out to be more robust. It seems to natural, since the control region in the interior control is larger. However, in the case of a single string or even for membranes, these control mechanisms are equivalent.

II. AN ELASTIC STRING CONTROLLED FROM ONE END

Consider the system

\[
\begin{align*}
\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} &= 0, & (t, x) \in \mathbb{R} \times [0, \ell], \\
\frac{\partial u}{\partial t}(t, \ell) &= 0, & u(t, 0) = v(t) & t \in \mathbb{R}, \\
u(0, x) &= u_0(x), & u_t(0, x) = u_1(x) & x \in [0, \ell].
\end{align*}
\]

This system describes the motion of an elastic homogeneous string controlled from the end $x = 0$ by means of a function $v$ that determines the displacement of that point. The functions $u_0$ and $u_1$ are the initial deformation and velocity of the string, respectively. We call the pair $(u_0, u_1)$ the initial state of the string.
Let us recall that for every $T > 0$ the equation (1) is well posed for initial states in $L^2(0, \ell) \times H^{-1}(0, \ell)$, and $v \in L^2(0, T)$; Given $(u_0, u_1) \in L^2(0, \ell) \times H^{-1}(0, \ell)$ and $v \in L^2(0, T)$ there exists a unique solution of (1) that satisfies

$$u \in C([0, T] : L^2(0, \ell)) \cap C^1([0, T] : H^{-1}(0, \ell)).$$

Given $T > 0$, we are interested in the possibility of driving the vibrations of the string to rest in time $T$ by means of the control $v$. That is, given an initial state $(u_0, u_1)$, to determine whether it is possible to find a function $v \in L^2(0, T)$ such that the solution of (1) satisfies

$$u(T, \cdot) = u_T(T, \cdot) = 0.$$

When such a function $v$ exists, it is said that the initial state $(u_0, u_1)$ is controllable in time $T$.

This kind of controllability problem may be considered for different equations and systems. When all the initial states from a subspace are controllable in time $T$ is said that the subspace is controllable in time $T$. In particular, when there exists a controllable subspace that coincides or in dense in a space where the equation is well posed it is said that the equation is exactly or approximately controllable, respectively.

The equation (1) turns out to be exactly controllable in any time $T \geq 2\ell$. This assertion may be proved using different techniques. Given an initial state $(u_0, u_1)$ it is possible, using D’Alembert formula for the representation of solutions of the wave equation, to compute explicitly the control $v$ that drives the initial state to rest (see, e.g., Chapter I in [K]). The method of moments can be also successfully applied (see, e.g., [AI], where this method is applied to study the controllability problem in multiple situations).

We shall focus, however, on the so-called Hilbert Uniqueness Method, due to J.-L. Lions. The key point of this method is the fact that the exact controllability in time $T$ is equivalent to the existence of a constant $C > 0$ such that the inequality

$$\int_0^T \left| \phi_x(t, \ell) \right|^2 dt \geq CE_{\phi}$$

holds for any solution $\phi$ of the homogeneous equation

$$\left\{ \begin{array}{ll}
\phi_{tt} - \phi_{xx} = 0 & (t, x) \in \mathbb{R} \times [0, \ell], \\
\phi(t, \ell) = \phi(t, 0) = 0 & t \in \mathbb{R},
\end{array} \right. \tag{3}$$

where $E_{\phi}$ is the energy of the solution $\phi$, a conserved quantity:

$$E_{\phi} = \frac{1}{2} \int_0^\ell \left( |\phi_x(t, x)|^2 + |\phi_t(t, x)|^2 \right) dx.$$

The inequality (2) is called observability inequality, as it indicates the fact that the whole energy of the solution $\phi$ may be estimated observing the trace $\phi_x(t, \ell)$ of the solution $\phi$ at point $x = \ell$.

This method has been widely used to study the controllability of numerous controlled systems. We recommend the book [Li2] for more details and examples of the application of HUM.

As we have mentioned before, the inequality (2) is true for the solutions of the equation (3). This fact may be proved using the D’Alembert formula for the representation of the solutions of (3) or by means of the multipliers method (see, e.g., [K]). The simplest way is, however, to use the Fourier series expansion of the solutions. We describe briefly the idea of the proof.

The solution of (3) may be expressed as

$$\phi(t, x) = \sum_{n \in \mathbb{Z}} a_n e^{i\lambda_n t} \sin \lambda_n x, \tag{4}$$

where $(\lambda_n)$ is the sequence formed by the square roots of the eigenvalues of the string with Dirichlet boundary conditions:

$$\lambda_n = \frac{n\pi}{\ell} \text{sgn } n, \quad n \in \mathbb{Z},$$

and
\[ a_n = \frac{1}{2} \left( \phi_{0,|n|} + i \lambda_n \phi_{1,|n|} \right), \quad n \in \mathbb{Z}_*, \]
where \( (\phi_{0,n}), (\phi_{1,n}) \) are the sequences of Fourier coefficients of \( \phi(0,.), \phi_t(0,.), \) respectively, in the orthogonal basis \( (\sin \lambda_n x) \) of \( L^2(0, \ell) \):
\[
\phi(0,x) = \sum_{n \in \mathbb{N}} \phi_{0,n} \sin \lambda_n x, \quad \phi_t(0,x) = \sum_{n \in \mathbb{N}} \phi_{1,n} \sin \lambda_n x.
\]
Using formula (4) the inequality (2) becomes
\[
\int_0^T \left| \sum_{n \in \mathbb{Z}_*} a_n e^{i \lambda_n t} \right|^2 dt \geq C \sum_{n \in \mathbb{N}} |a_n|^2.
\]
Now, if \( T \geq 2 \ell \) we have
\[
\int_0^T \left| \sum_{n \in \mathbb{Z}_*} a_n e^{i \lambda_n t} \right|^2 dt \geq \int_0^{2\ell} \left| \sum_{n \in \mathbb{Z}_*} a_n e^{i \lambda_n t} \right|^2 dt = C \sum_{n \in \mathbb{N}} |a_n|^2
\]
(here we have used the fact that the functions \( (e^{i \lambda_n t}) \) are orthogonal on the interval \( (0, \ell) \)) and thus, the inequality (2) is proved for \( T \geq 2 \ell \). When \( T < 2 \ell \) an inequality like (2) does not hold. Indeed, due to the finite speed (equal to one in this case) of propagation of the waves along the string, it is possible to construct for any \( T < 2 \ell \) a non-zero solution \( \phi \) of (3) such that
\[
\phi_x(t, \ell) = 0 \quad \text{in} \quad (0,T),
\]
and then
\[
\int_0^T |\phi_x(t, \ell)|^2 dt = 0.
\]
Since the energy \( E_\phi \) of such solution is different from zero, inequality (2) cannot be true.

In the proof of inequality (2) we have given, it has been used in an essential way the orthogonality of the functions \( (e^{i \lambda_n t}) \). In some problems however, we do not have this kind of property (we shall see some examples below). Nevertheless, in such cases it is sometimes still possible to proceed in a similar way using the Ingham inequality (see [In]) whenever between the numbers \( \lambda_n \) there is certain uniform gap:
\[
\lambda_{n+1} - \lambda_n \geq \gamma > 0
\]
for every \( n \).

Unfortunately, in the problems of simultaneous control of strings or control of networks this is not the case.

III. SIMULTANEOUS CONTROL OF TWO STRINGS

Now we consider a similar problem: two strings of lengths \( \ell_1 \) and \( \ell_2 \) to be controlled from one extreme, but with an additional difficulty: the control function should be the same for both strings. In some sense, this problem is equivalent to the controllability problem of a network formed by three strings coupled at one node and controlled from one of the exterior nodes of the network. From the technical point of view, the main difficulty of this problem consists in the fact that between the elements of the sequence of the corresponding eigenvalues there is no uniform gap. This feature implies that the system is not exactly controllable in any time and so, it is necessary to restrict ourselves to prove the controllability for small subspaces of initial states of the strings.
The simultaneous controllability problem for two strings was implicitly studied in [JTZ2]. Later on, in [TW] and
[AT] an essentially complete solution was given. The results in [TW] are based on a generalization of the Ingham
inequality proved in [JTZ2]. This technique, however, allows to prove the controllability in a time larger than the
minimal one. In [AT], using the method of moments, controllability results in the minimal time are obtained. The
solution that we present here is based on completely elementary arguments and, in addition, provides a more precise
information than the other techniques we have mentioned.

The system, which describes the simultaneous control of two strings is
\[
\begin{aligned}
    u^i_{tt} - u^i_{xx} &= 0 & (t, x) &\in \mathbb{R} \times [0, \ell_i], \\
    u^i(t, \ell_i) &= 0, & u^i(t, 0) &= v(t) & t &\in \mathbb{R}, \\
    u^i(0, x) &= u^i_0(x), & u^i_1(0, x) &= u^i_1(x) & x &\in [0, \ell_i],
\end{aligned}
\]
for \(i = 1, 2\). In this case, simultaneous indicates the fact that the same control \(v\) is applied to both strings. This kind
of problem was first considered by Russell [Ru3]. The chapter 5 in [Li2] is devoted to the study of simultaneous
control problems in different situations.

For every \(T > 0\) the system (6) is well posed for initial states from \((u^i_0, u^i_1) \in L^2(0, \ell_i) \times H^{-1}(0, \ell_i), i = 1, 2\)
and \(v \in L^2(0, T)\): there exists a unique solution that satisfies
\[
u^i \in C([0, T] : L^2(0, \ell_i)) \cap C^1([0, T] : H^{-1}(0, \ell_i)),
\]
for \(i = 1, 2\).

When \(v \equiv 0\) the system (6) becomes
\[
\begin{aligned}
    \phi^i_{tt} - \phi^i_{xx} &= 0 & (t, x) &\in \mathbb{R} \times [0, \ell_i], \\
    \phi^i(t, \ell_i) &= \phi^i(t, 0) = 0 & t &\in \mathbb{R}, \\
    \phi^i(0, x) &= \phi^i_0(x), & \phi^i_1(0, x) &= \phi^i_1(x) & x &\in [0, \ell_i],
\end{aligned}
\]
with \(i = 1, 2\). Let us remark that the system (7) is formed by two uncoupled wave equations with Dirichlet
homogeneous boundary conditions. Both equations are also well posed for initial states \((\phi^i_0, \phi^i_1) \in H^1_0(0, \ell_i) \times
L^2(0, \ell_i)\) and the corresponding solutions are expressed by the formula
\[
\phi^i(t, x) = \sum_{n \in \mathbb{N}} \left( \phi^i_{0,n} \cos \sigma_n^i t + \frac{\phi^i_{1,n}}{\sigma_n^i} \sin \sigma_n^i t \right) \sin \sigma_n^i x,
\]
for \(i = 1, 2\), where \((\sigma_n^i)\) is the sequence formed by the square roots of the eigenvalues of the string \(e_i\):
\[
\sigma_n^i = \frac{n\pi}{\ell_i}, \quad n \in \mathbb{N},
\]
and \((\phi^i_{0,n}), (\phi^i_{1,n})\) are the sequences of Fourier coefficients of \(\phi^i_0, \phi^i_1\), respectively, in the orthogonal basis \((\sin \sigma_n^i x)\)
of \(L^2(0, \ell_i)\):
\[
\phi^i_0(x) = \sum_{n \in \mathbb{N}} \phi^i_{0,n} \sin \sigma_n^i x, \quad \phi^i_1(x) = \sum_{n \in \mathbb{N}} \phi^i_{1,n} \sin \sigma_n^i x,
\]
for \(i = 1, 2\).

The simultaneous controllability problem in time \(T\) consists in describing those initial states \((u^i_0, u^i_1), i = 1, 2,\)
for system (6) such that there exists \(v \in L^2(0, T)\) such that the solutions \(u^1, u^2\) of (6) satisfy
\[
u^i(T, x) = u^i_1(T, x) = 0, \quad i = 1, 2,
\]
for \(x \in [0, \ell_i]\).
Let us remark that, though the system (7) is formed by two uncoupled equations, the fact that the same control function is used in both equations, makes it impossible to consider the equations separately. Indeed, if we apply the HUM method it turns out that the observability inequality associated to (6) is

$$\int_0^T |\phi_x^1(t,0) + \phi_x^2(t,0)|^2 dt \geq \sum_{i=1,2} \sum_{n \in \mathbb{N}} (e_{i,n}^2)((\sigma_n \phi_{0,n}^i)^2 + (\phi_{1,n}^i)^2)$$

(9)

for the solutions of (7). If there exist sequences of positive numbers \((c_{i,n}^0)\), \(i = 1, 2\) such that (9) holds for any solutions \(\phi^1, \phi^2\) of (7) with initial states \((\phi^1_0, \phi^1_1) \in Z^1 \times Z^1\), \((\phi^2_0, \phi^2_1) \in Z^2 \times Z^2\), respectively, then all the initial states \((u_0^i, u_1^i), i = 1, 2\), satisfying

$$\sum_{n \in \mathbb{N}} \frac{1}{(c_{i,n}^0)^2}(u_0^i)^2 + \sum_{n \in \mathbb{N}} \frac{1}{(c_{i,n}^1)^2}(u_1^i)^2 < \infty$$

are controllable in time \(T\). By \(Z^1\) and \(Z^2\) we have denoted the space of all finite linear combinations of the functions \(\sin \sigma_n x\) and \(\sin \sigma_n^2 x\), respectively.

The main observability result for the solutions of (7) is

**Theorem 1:** Let \(T^* = 2(\ell_1 + \ell_2)\). The following inequalities hold

$$\int_0^{T^*} |\phi_x^1(t,0) + \phi_x^2(t,0)|^2 dt \geq \ell_1 \sum_{n \in \mathbb{N}} (\sin \sigma_n^2 \ell_2)(\sin \sigma_n^1 \ell_2) \left(\sum_{i=1,2} \frac{1}{(c_{i,n}^0)^2}(u_0^i)^2 + \frac{1}{(c_{i,n}^1)^2}(u_1^i)^2\right),$$

(10)

$$\int_0^{T^*} |\phi_x^1(t,0) + \phi_x^2(t,0)|^2 dt \geq \ell_2 \sum_{n \in \mathbb{N}} (\sin \sigma_n^1 \ell_1)(\sin \sigma_n^2 \ell_1) \left(\sum_{i=1,2} \frac{1}{(c_{i,n}^0)^2}(u_0^i)^2 + \frac{1}{(c_{i,n}^1)^2}(u_1^i)^2\right),$$

(11)

for every solution of (7) with initial states \((\phi_0^0, \phi_0^1) \in Z^1 \times Z^1\).

**Proof:** We shall prove the second inequality. The first one may be obtained in an analogous way.

We define the linear operator \(\ell_1^-\), which acts on a continuous function \(f\) by the formula

$$\ell_1^- f(t) = \frac{f(t + \ell_1) - f(t - \ell_1)}{2}.\tag{10}$$

This operator is continuous from \(L^2(0, T^*)\) to \(L^2(\ell_1, T^* - \ell_1)\):

$$\int_0^{T^*} |f(t)|^2 dt \geq \int_{\ell_1}^{T^* - \ell_1} |\ell_1^- f(t)|^2 dt,\tag{11}$$

for every \(f\) for which the integrals are defined.

Let us observe that, due to the \(2\ell_1\)-periodicity in time of the solutions of (7) it holds \(\ell_1^- \phi_x^1(t,0) = 0\). Then, from (11) we obtain

$$\int_0^{T^*} |\phi_x^1(t,0) + \phi_x^2(t,0)|^2 dt \geq \int_{\ell_1}^{T^* - \ell_1} |\ell_1^- \phi_x^1(t,0) + \ell_1^- \phi_x^2(t,0)|^2 dt = \int_{\ell_1}^{T^* - \ell_1} |\ell_1^- \phi_x^2(t,0)|^2 dt.\tag{12}$$
On the other hand, $\psi = \ell_1^{-1} \phi^2$ is a solution of the equation
\[ \psi_{tt} - \psi_{xx} = 0 \]
in $\mathbb{R} \times [0, \ell_1]$ and consequently, using the results of the previous section we get
\[ \int_{\ell_1}^{T-\ell_1} |\psi_x(t,0)|^2 dt \geq 4E_\psi. \] (13)

Taking into account that $\psi(x,0) = \ell_1^{-1} \phi^2_x(t,0)$, from (12) and (13) it follows that
\[ \int_{0}^{T} |\phi^1_x(t,0) + \phi^2_x(t,0)|^2 dt \geq 4E_{\ell_1^{-1} \phi^2}. \] (14)

It just remains to calculate the energy $E_{\ell_1^{-1} \phi^2}$. From formula (8) we obtain
\[ \ell_1^{-1} \phi^2(t, x) = \sum_{n \in \mathbb{N}} (\phi_{0,n}^2 \ell_1^{-1} \cos \sigma_n^2 t + \phi_{1,n}^2 \ell_1^{-1} \sin \sigma_n^2 t) \sin \sigma_n^2 x. \] (15)

Now, in view of the relations
\[ \ell_1^{-1} \cos \sigma_n^2 t = \frac{1}{2} \left( \cos \sigma_n^2 (t + \ell_1) - \cos \sigma_n^2 (t - \ell_1) \right) = - \sin \sigma_n^2 \ell_1 \sin \sigma_n^2 t, \]
\[ \ell_1^{-1} \sin \sigma_n^2 t = \frac{1}{2} \left( \sin \sigma_n^2 (t + \ell_1) - \sin \sigma_n^2 (t - \ell_1) \right) = \sin \sigma_n^2 \ell_1 \cos \sigma_n^2 t, \]
the equality (15) becomes
\[ \ell_1^{-1} \phi^2(t, x) = \sum_{n \in \mathbb{N}} \sin \sigma_n^2 \ell_1 \left( \frac{\phi_{0,n}^2}{\sigma_n^2} \ell_1^{-1} \cos \sigma_n^2 t - \phi_{1,n}^2 \ell_1^{-1} \sin \sigma_n^2 t \right) \sin \sigma_n^2 x. \]

On the other hand, it is easy to see that the energy is given by
\[ E_{\ell_1^{-1} \phi^2} = \frac{\ell_2}{4} \sum_{n \in \mathbb{N}} \left( \sin \sigma_n^2 \ell_1 \right)^2 \left( (\sigma_n^2 \phi_{0,n})^2 + (\phi_{1,n}^2)^2 \right). \]

Thus, it suffices to place the latter expression in (14) to obtain the inequality stated in the theorem. \( \square \)

A. **Identification of controllable spaces**

The goal of this subsection is to identify subspaces of controllable initial states for the system (6) in time $T \geq 2(\ell_1 + \ell_2)$ on the basis of Theorem 1. As we have said above, controllable subspaces of initial states are necessary smaller than the space where the system is well posed.

An easily identifiable subspace is the one formed by the finite linear combinations of the eigenfunctions. When all the finite linear combinations of the eigenfunctions are controllable in time $T$ the system is said to be **spectrally controllable**. Clearly, spectral controllability is a particular case of approximate controllability. It takes place:

**Proposition 1**: The system (6) is spectrally controllable in some time $T \geq 2(\ell_1 + \ell_2)$ (and then in any time $T \geq 2(\ell_1 + \ell_2)$) if, and only if, the ratio $\frac{\ell_1}{\ell_2}$ is an irrational number.

**Proof**: If $\frac{\ell_1}{\ell_2}$ is an irrational number, the coefficients $\sin \sigma_n^1 \ell_2$, $\sin \sigma_n^2 \ell_1$, $n \in \mathbb{N}$, appearing in the inequalities from Proposition 1 are all different from zero. Indeed, if $\sin \sigma_n^2 \ell_1 = 0$ for some $n$, then there exists $k \in \mathbb{N}$ such that
\[ \frac{n\pi \ell_2}{\ell_1} = k \pi. \]
that is, \( \frac{\ell_1}{\ell_2} = \frac{n}{k} \in \mathbb{Q} \). Then, the initial states \((u_0^i, u_1^i), i = 1, 2\), which satisfy

\[
\sum_{n \in \mathbb{N}} \frac{1}{(\sin \sigma_n^1 \ell_2)^2} (u_{0,n}^1)^2 + \sum_{n \in \mathbb{N}} \frac{1}{(\sin \sigma_n^1 \ell_2)^2} (u_{1,n}^1)^2 < \infty,
\]

\[
\sum_{n \in \mathbb{N}} \frac{1}{(\sin \sigma_n^2 \ell_1)^2} (u_{0,n}^2)^2 + \sum_{n \in \mathbb{N}} \frac{1}{(\sin \sigma_n^2 \ell_1)^2} (u_{1,n}^2)^2 < \infty,
\]

are controllable in time \( T \geq 2(\ell_1 + \ell_2) \).

In particular, the states \((\phi_0^1, \phi_1^1) \in Z^1 \times Z^1, (\phi_0^2, \phi_1^2) \in Z^2 \times Z^2\) are controllable.

We shall now see that the condition \( \frac{\ell_1}{\ell_2} \notin \mathbb{Q} \) is also necessary for the approximate controllability. If \( \frac{\ell_1}{\ell_2} = \frac{n}{k} \) with \( n, k \in \mathbb{N} \) then, for every \( p \in \mathbb{N} \) the functions

\[
\phi^1(t, x) = \sin \frac{pn \pi t}{\ell_1} \sin \frac{pn \pi x}{\ell_1},
\]

\[
\phi^2(t, x) = -\sin \frac{pk \pi t}{\ell_2} \sin \frac{pk \pi x}{\ell_2},
\]

are solutions of (7) and satisfy

\[
\phi^1(t, 0) + \phi^2_x(t, 0) = 0.
\]

Therefore, the system (6) is not approximately controllable and, in particular, is not spectrally controllable in any time.

For the further identification of controllable initial states for the system (6) on the basis of Theorem 1 we shall need some definitions from Number Theory. For \( \eta \in \mathbb{R} \) we denote by \( |||\eta||| \) the distance from \( \eta \) to the set \( \mathbb{Z} \) of integer numbers:

\[
|||\eta||| = \min\{ x \in \mathbb{R} : \eta - x \in \mathbb{Z} \}.
\]

**Proposition 2:** If \( \frac{\ell_1}{\ell_2} \) is irrational, all the initial states \((u_0^1, u_1^1), (u_0^2, u_1^2)\) satisfying

\[
\sum_{n \in \mathbb{N}} \frac{1}{|||\frac{\ell_2}{\ell_1}|||^2} (u_{0,n}^1)^2 + \sum_{n \in \mathbb{N}} \frac{1}{|||\frac{\ell_2}{\ell_1}|||^2} (u_{1,n}^1)^2 < \infty,
\]

\[
\sum_{n \in \mathbb{N}} \frac{1}{|||\frac{\ell_2}{\ell_1}|||^2} (u_{0,n}^2)^2 + \sum_{n \in \mathbb{N}} \frac{1}{|||\frac{\ell_2}{\ell_1}|||^2} (u_{1,n}^2)^2 < \infty,
\]

are controllable in any time \( T \geq 2(\ell_1 + \ell_2) \).

**Proof:** Let us observe that for any \( x \in \mathbb{R} \)

\[
2|||\frac{x}{\pi}||| \leq |\sin x| \leq \pi|||\frac{x}{\pi}|||
\]

Then,

\[
2|||\frac{\ell_2}{\ell_1}||| \leq |\sin \sigma_1^1 \ell_2| \leq \pi|||\frac{\ell_2}{\ell_1}|||,
\]

\[
2|||\frac{\ell_1}{\ell_2}||| \leq |\sin \sigma_2^2 \ell_1| \leq \pi|||\frac{\ell_1}{\ell_2}|||.
\]

Therefore, the relations (18)-(19) are equivalent to (16)-(17).

In order to characterize subspaces of controllable initial states for the system (6) it will be sufficient to estimate the sequences \( |||n_2^1|||, |||n_2^2|||, n \in \mathbb{N} \).
A natural way of getting additional information is the following: let \( \rho : \mathbb{R} \rightarrow \mathbb{R}_+ \) be an increasing function

\[
\Psi_\rho = \left\{ x \in \mathbb{R}_+ : \liminf_{n \to \infty} ||nx|| \rho(n) > 0 \right\}.
\]

Then, if \( \ell_1, \ell_2 \in \Psi_\rho \) the inequalities

\[
\sum_{n \in \mathbb{N}} \rho(n)(u_{0,n}^1)^2 + \sum_{n \in \mathbb{N}} \rho(n)(u_{1,n}^1)^2 < \infty, \tag{21}
\]

\[
\sum_{n \in \mathbb{N}} \rho(n)(u_{0,n}^2)^2 + \sum_{n \in \mathbb{N}} \rho(n)(u_{1,n}^2)^2 < \infty, \tag{22}
\]

guarantee the controllability of \((u_{0}^1, u_{1}^1), (u_{0}^2, u_{1}^2)\).

However, in such a general setting, the problem turns out to be extremely difficult. That is why we restrict ourselves to considering \( \rho(x) = x^\alpha \) with \( \alpha > 0 \). This choice is motivated by two reasons. The first one is that such choice of \( \rho \) leads to the identification of spaces of controllable initial states of the form

\[
(u_{0}^i, u_{1}^i) \in \hat{H}^\alpha(0, \ell_i) \times \hat{H}^{\alpha-1}(0, \ell_i),
\]

where

\[
\hat{H}^{\varepsilon}(0, \ell_i) = \left\{ u(x) = \sum_{n \in \mathbb{N}} u_n \sin \alpha_n x : \sum_{n \in \mathbb{N}} n^{2\varepsilon}(u_n)^2 < \infty \right\}.
\]

Let us note that \( \hat{H}^{\varepsilon}(0, \ell_i) \) is the Sobolev space \( H^{\varepsilon}(0, \ell_i) \) with some additional boundary conditions. In particular, \( \hat{H}^1(0, \ell_i) = H_0^1(0, \ell_i) \) and \( \hat{H}^0(0, \ell_i) = L^2(0, \ell_i) \).

The second reason for this choice of \( \rho \) is that the problem of describing the sets

\[
\Psi_\alpha := \Psi_{x^\alpha} = \left\{ x \in \mathbb{R}_+ : \liminf_{n \to \infty} ||nx|| n^{\alpha} > 0 \right\},
\]

is a classical (and difficult) problem in Number Theory. In [La] and [C] it may be found extensive information on this theme.

**Positive results**

The following facts are known

1) For every \( \alpha > 0 \) the sets \( \Psi_\alpha \) have the property: if \( \xi \in \Psi_\alpha \) then \( \frac{1}{\xi} \in \Psi_\alpha \).

2) \( \Psi_1 \) coincides with the sets of irrational numbers \( \eta \in \mathbb{R} \) such that in their expansion in continuous fraction \([a_0, a_1, \ldots, a_n, \ldots]\) (see, e. g., [La], p. 6) the sequence \((a_n)\) is bounded. The set \( \Psi_1 \) is not denumerable and has Lebesgue measure equal to zero.

3) For every \( \varepsilon > 0 \) the complementary of the set \( \Psi_{1+\varepsilon} \) is of zero measure. This set is usually denoted by \( B_\varepsilon \subset \mathbb{R} \). As a consequence of the Roth theorem, the set \( B_\varepsilon \) contains all the algebraic irrational numbers, that is, those numbers that are root of polynomials of degree greater than one with integer coefficients.

Consequently we obtain

**Corollary 1:** a) If \( \frac{\ell_1}{\ell_2} \in B_\varepsilon \) then, the subspace of initial states

\[
(u_{0}^i, u_{1}^i) \in \hat{H}^{1+\varepsilon}(0, \ell_i) \times \hat{H}^{\varepsilon}(0, \ell_i),
\]

is controllable in any time \( T \geq 2(\ell_1 + \ell_2) \). In particular, if \( \frac{\ell_1}{\ell_2} \) is an algebraic irrational number, this subspace is controllable for any \( \varepsilon > 0 \).

b) If \( \frac{\ell_1}{\ell_2} \) admits a bounded expansion in continuous fraction then, the subspace of initial states

\[
(u_{0}^i, u_{1}^i) \in H_0^1(0, \ell_i) \times L^2(0, \ell_i),
\]
is controllable in any time $T \geq 2(\ell_1 + \ell_2)$.

**Negative results**

We describe now some results, which stand as counterpart of those provided by Proposition 2.

**Proposition 3:** Assume there exists a sequence $(n_k) \subset \mathbb{N}$ such that

$$|||n_k\frac{\ell_1}{\ell_2}|||\rho(n_k) \to 0 \quad \text{and} \quad |||n_k\frac{\ell_2}{\ell_1}|||\rho(n_k) \to 0, \quad k \to \infty.$$  

There, there exist initial states $(u_0^1, u_1^1), (u_0^2, u_1^2)$ satisfying (21)-(22), which are not controllable in any finite time $T$.

**Proof:** Let us recall that the fact that all the initial states satisfying (21)-(22) are controllable in time $T$ is equivalent to the inequalities

$$\int_0^T |\phi_1^1(t, 0) + \phi_2^2(t, 0)|^2 dt \geq C_1 \sum_{n \in \mathbb{N}} \frac{1}{\rho^2(n)} \left( \left( \frac{n\pi}{\ell_1} \phi_{0,n}^1 \right)^2 + \left( \phi_{1,n}^1 \right)^2 \right)$$

being true for every solution of (7) with initial states $(\phi_0^i, \phi_1^i) \in \mathbb{Z}^i \times \mathbb{Z}^i$, $i = 1, 2$.

We shall assume that

$$|||n_k\frac{\ell_1}{\ell_2}|||\rho(n_k) \to 0$$

and we shall prove that in this case the inequality (24) is impossible.

Indeed, from (25) it follows that, for every $k \in \mathbb{N}$, there exists $m_k \in \mathbb{N}$ such that

$$\left|n_k\frac{\ell_1}{\ell_2} - m_k\right| \rho(n_k) \to 0.$$ 

Then,

$$|\sigma_n^2 - \sigma_{m_k}^1| \rho(n_k) = \left|\frac{\pi n_k}{\ell_1} - \frac{\pi m_k}{\ell_2}\right| \rho(n_k) \to 0.$$  

On the other hand, if we replace in (24) the solutions

$$\phi_k^1(t, x) = \cos \sigma_{m_k}^1 t \sin \sigma_{m_k}^1 x,$$

$$\phi_k^2(t, x) = -\cos \sigma_{n_k}^2 t \sin \sigma_{n_k}^2 x,$$

we obtain

$$\int_0^T |\sigma_{m_k}^1 \cos \sigma_{m_k}^1 t - \sigma_{n_k}^2 \cos \sigma_{n_k}^2 t|^2 dt \geq C_2 \rho^{-2}(n_k)(\sigma_{n_k}^2)^2$$

and consequently

$$|\sigma_{n_k}^2 - \sigma_{m_k}^1|^2 \geq C \rho^{-2}(n_k)$$

(we have used here the inequality

$$\int_0^T |x \cos xt - y \cos yt|^2 dt \leq 4|x - y|^2 x^2 T,$$
for \( y \geq x \geq 1 \), which may be easily obtained, using, e.g., the mean value theorem).

Thus, from (27) it holds

\[
|\sigma_{n_k}^2 - \sigma_{m_k}^1| \rho(n_k) \geq C,
\]

what contradicts the property (26) of the sequences \((n_k)\) and \((m_k)\).

The first important consequence of Proposition 3 is based on the Dirichlet theorem: for every \( \alpha < 1 \), \( \xi \in \mathbb{R} \) and \( \varepsilon > 0 \) there exist an infinite number of values of \( n \) such that \( ||n \xi||n^\alpha < \varepsilon \) (see [C], Section I.5).

**Corollary 2:** For all the values \( \ell_1, \ell_2 \) of the lengths of the strings and any \( \alpha < 1 \) there exist initial states

\[
(u_0^i, u_1^i) \in \tilde{H}^\alpha(0, \ell_i) \times \tilde{H}^{\alpha-1}(0, \ell_i), \quad i = 1, 2,
\]

there exist initial states, which are not controllable in any finite time \( T \). In particular, there exist non controllable initial states from \( L^2(0, \ell_i) \times H^{-1}(0, \ell_i) \), i.e., the system (6) is not exactly controllable in any finite time.

The following result of negative character is based on a construction due to Liouville. Let us consider the series

\[
\xi = \sum_{k \in \mathbb{N}} 10^{-a_k},
\]

where \((a_k)\) is an increasing sequence of positive integers. Then, for any \( p \in \mathbb{N} \),

\[
||\xi 10^{a_p} - m|| = 10^{a_p} \sum_{k > p} 10^{-a_k} < 10^{a_p - a_{p+1}}.
\]

Now assume that \( \rho : \mathbb{R} \rightarrow \mathbb{R}_+ \) is an increasing function. Fix \( \varepsilon > 0 \) and choose a sequence \((a_k)\) verifying

\[
10^{a_k - a_{k+1}} < \frac{\varepsilon}{\rho(10^{a_k})},
\]

what is equivalent to

\[
a_{k+1} > a_k + \lg \frac{\rho(10^{a_k})}{\varepsilon}.
\]

Then, for the numbers \( n_p = 10^{a_p}, \ p \in \mathbb{N} \), it holds

\[
|||n_p \xi||| \rho(n_p) < \varepsilon.
\]

Thus, it is possible to construct real numbers that are approximated by rationals faster than any given order \( \rho \). From Proposition 3 we obtain

**Corollary 3:** For any increasing function \( \rho : \mathbb{R} \rightarrow \mathbb{R}_+ \), it is possible to find values \( \ell_1, \ell_2 \) of the lengths of the strings such that there exist initial data from the subspace defined by (21)-(22), which are not controllable in any finite time \( T \). In other words, the space of controllable initial states for the system may be arbitrarily small.

**IV. SIMULTANEOUS INTERIOR CONTROL**

In this section we study the problem of controlling simultaneously two strings with different lengths and densities from an interior region.

After performing the change of variables \( x \rightarrow \ell_1 x, \ x \rightarrow \ell_2 x \) in the equations of system (6) for the simultaneous control of two strings from one of the ends we get

\[
\begin{align*}
\ell_k^2 u_{tt}^k - u_{xx}^k & = 0 \quad \text{in } \mathbb{R} \times [0, 1], \quad k = 1, 2, \\
u_k(., 0) & = v, \quad u_k(., 1) = 0 \quad \text{in } \mathbb{R}, \\
u_k(0, .) & = u_0^k, \quad u_t^k(0, .) = u_t^k \quad \text{in } [0, 1].
\end{align*}
\]

Thus, the problem of simultaneous controllability problem for two string with lengths \( \ell_1 \) and \( \ell_2 \) from one of the ends of the strings can also be viewed as a simultaneous control problem for two strings with length equal to one and densities \( \ell_1 \) and \( \ell_2 \).

This suggests to study the similar problem in the case when the control acts from an interior region of the strings. This problem is studied in the subsection 1. When the lengths of the strings coincide and the control acts on the whole string, it is possible to control them in any arbitrarily small time. This is true, even in the multidimensional case. We address this problem in the subsection 2.
A. Simultaneous interior control of two strings

1) Problem formulation: Let $\ell_1$ and $\ell_2$ be positive numbers and $\omega$ an interval contained in $(0, \ell_1) \cap (0, \ell_2)$. We consider the system

\[
\begin{cases}
\rho_k^2 u_{tt}^k - u_{xx}^k + f(x) \chi_\omega = 0 & \text{in } \mathbb{R} \times [0, \ell_k], \\
u^k(.,0) = u^k(., \ell_k) = 0 & \text{in } \mathbb{R}, \\
u^k(0,.) = u^k_0, \quad u^k_1(0,.) = u^k_1 & \text{in } [0, \ell_k],
\end{cases}
\]  

(30)

where $f \in L^2_{\text{loc}}(\mathbb{R}^2)$ and $\chi_\omega$ is the characteristic function of the interval $\omega$.

This system describes the motion of two strings $e_1$ and $e_2$ with lengths $\ell_1$, $\ell_2$ and densities $\rho_1, \rho_2$, respectively, which are controlled by means of a common action localized on the interval $\omega$.

The system (30) is well posed for initial data

$$(u^k_0, u^k_1) \in W_k := H^1_0(0, \ell_k) \times L^2(0, \ell_k), \quad k = 1, 2.$$  

We study the following control problem for (30): Given $T > 0$, to determine for which initial states $(u^k_0, u^k_1), k = 1, 2$, a control function $f$ may be chosen such that the solution of (30) satisfies

$$u^k(T,.) = u^k_1(T,.) = 0, \quad k = 1, 2.$$  

We shall say that the system (30) is exactly controllable in time $T$ if all the initial states $(u^k_0, u^k_1) \in W_k$, $k = 1, 2$, are controllable in time $T$.

Proceeding as before, the application of HUM guarantees that the system (30) is exactly controllable in time $T$ if, and only if, there exists a constant $C > 0$ such that

$$C \int_0^T \int_\omega |\phi^1(t,x) + \phi^2(t,x)|^2 dtdx$$

\[
\geq ||(\phi^1_0, \phi^1_1)||^2_{L^2(0,T) \times H^{-1}} + ||(\phi^2_0, \phi^2_1)||^2_{L^2(0,T) \times H^{-1}},
\]

(31)

for all the solutions $\phi^1, \phi^2$ of the homogeneous equations

\[
\begin{cases}
\rho_k^2 \phi_{tt}^k - \phi_{xx}^k = 0 & \text{in } \mathbb{R} \times [0, \ell_k], \quad k = 1, 2, \\
\phi^k(.,0) = \phi^k(., \ell_k) = 0 & \text{in } \mathbb{R}, \\
\phi^k(0,.) = \phi^k_0, \quad \phi^k_1(0,.) = \phi^k_1 & \text{in } [0, \ell_k].
\end{cases}
\]  

(32)

The solutions of (32) are given by the formula

$$\phi^k(t,x) = \sum_{n \in \mathbb{N}} (\phi^k_{0,n} \cos \frac{n\pi t}{\rho_k \ell_k} + \frac{\rho_k \ell_k}{n\pi} \phi^k_{1,n} \sin \frac{n\pi t}{\rho_k \ell_k}) \sin \frac{n\pi x}{\ell_k},$$  

(33)

where $(\phi^k_{0,n}), (\phi^k_{1,n})$ are the sequences of Fourier coefficients of $\phi^k_0, \phi^k_1$, respectively, in the basis $(\sin \frac{n\pi x}{\ell_k})_{n \in \mathbb{N}}$ of $L^2(0, \ell_k)$.

If we denote

$$a^k_n = \frac{1}{2} \left( \phi^k_{0,|n|} + \frac{i \rho_k \ell_k}{n\pi} \phi^k_{1,|n|} \right) , \quad k = 1, 2, \quad n \in \mathbb{Z}_a,$$

the formula (33) may be rewritten as

$$\phi^k(t,x) = \sum_{n \in \mathbb{Z}_a} a^k_n \cos \frac{n\pi x}{\ell_k} \sin \frac{n\pi x}{\ell_k},$$
With these notations the inequality (31) becomes
\[ C \int_\omega \int_0^T \left| \sum_{n \in \mathbb{Z}_1} a_n e^{i n t \ell_1} \sin \frac{n \pi x}{\ell_1} + a_n^2 e^{i n t \ell_2} \sin \frac{n \pi x}{\ell_2} \right|^2 dt dx \]
\[ \geq \sum_{n \in \mathbb{N}} (|a_n^1|^2 + |a_n^2|^2), \]  \hspace{1cm} (34)
for any finite sequences \((a_n^1)_{n \in \mathbb{Z}_1}, (a_n^2)_{n \in \mathbb{Z}_2}\) satisfying \(a_{1-n}^1 = \overline{a_n^1}, a_{2-n}^2 = \overline{a_n^2}\).

Obviously, the inequality (34) is impossible if \(\ell_1 = \ell_2\) and \(\rho_1 = \rho_2\). Indeed, it would suffice to choose, e. g., \(a_1^1 = -a_1^2 \neq 0\) and \(a_n^1 = -a_n^2 = 0\) if \(n \neq \pm 1\), to see that in this case (34) is not verified. Our goal is to prove that the inequality (34) holds whenever \(\rho_1 \neq \rho_2\) if \(T\) is sufficiently large. This is the object of the next subsection.

2) Control of strings with different densities: Assume \(\rho_1 \neq \rho_2\). It holds:

**Theorem 2:** If \(T > T_0 := 2 \max(\rho_1 \ell_1, \rho_2 \ell_2)\) then the inequality (34) is true for all the finite complex sequences \((a_n^1)_{n \in \mathbb{Z}_1}, (a_n^2)_{n \in \mathbb{Z}_2}\) that verify \(a_{1-n}^1 = \overline{a_n^1}, a_{2-n}^2 = \overline{a_n^2}\).

**Corollary 4:** The strings \(e_1\) and \(e_2\) are simultaneously exactly controllable in any time \(T > T_0\) if, and only if, \(\rho_1 \neq \rho_2\).

**Remark 1:** This result shows an important difference between the control from one end and the control from an arbitrarily small interior region of the strings. We recall that according to the Corollary 1, all the initial states from \((H_0^1(0,1) \times L^2(0,1))^2\) for the system (29) are controllable in time \(T \geq 2(\ell_1 + \ell_2)\) if, and only if, the ratio \(\frac{\ell_1}{\ell_2}\) belongs to the set \(\mathcal{F}\), whose measure is equal to zero. Besides, the exact controllability of (29) is never reached, regardless of the values of \(\ell_1\) and \(\ell_2\). This shows that the controllability from an interior region is a much more robust property than the controllability from one end.

To prove Theorem 2 we follow the scheme used in the proof of Theorem 1 concerning the simultaneous control of two strings from one end. The idea is once again quite simple: we construct a continuous operator \(B : L^2(\omega \times (0,T)) \rightarrow L^2(\omega' \times (0,T'))\) such that if \(\phi^1, \phi^2\) are solutions of (32) then \(B \phi^1 = 0\). Moreover, there exists a constant \(C > 0\) such that
\[ C \int_\omega \int_0^T |B \phi^2|^2 dx dt \geq \| (\phi_0^2, \phi_1^2) \|^2_{L^2 \times H^{-1}}. \]
Then the main step is
\[ C \int_\omega \int_0^T |\phi^1 + \phi^2|^2 dx dt \]
\[ \geq C \int_\omega \int_0^T |B \phi^2|^2 dx dt \geq \| (\phi_0^2, \phi_1^2) \|^2_{L^2 \times H^{-1}}. \]

The inequality
\[ C \int_\omega \int_0^T |\phi^1 + \phi^2|^2 dx dt \geq \| (\phi_0^1, \phi_1^1) \|^2_{L^2 \times H^{-1}} \]
is obtained in an analogous way.

We fix \(\omega_1 \in \mathbb{R}\) and define for any \(\alpha \in \mathbb{R}\), the linear operator \(B_\alpha\) that acts on a function \(\phi(t,x)\) according to the formula
\[ B_\alpha \phi(t,x) := \phi(t + 2a(x - \omega_1), x + a(x - \omega_1)) \]
\[ - \phi(t + a(x - \omega_1), x + 2a(x - \omega_1)) \]
\[ - \phi(t + a(x - \omega_1), x + a(x - \omega_1)). \]

Let us remark that, as \(\omega_1 < \omega_2\) and \(T > 0\), it is possible to choose \(\hat{\omega}_2 > \omega_1\) such that
\[ \hat{\omega}_2 < \frac{\omega_2 + 2a\omega_1}{1 + 2a} \quad \text{and} \quad \hat{T} := T - 2a(\hat{\omega}_2 - \omega_1) > 0. \]  \hspace{1cm} (35)

1We recall that is \(\mathcal{F}\) is formed by those irrational numbers in whose expansion in continuous fraction \([a_0, a_1, ..., a_n, ...]\) the sequence \((a_n)\) is bounded.
Proposition 4: If \( \dot{\omega}_2 \) and \( \dot{T} \) satisfy (35) then the operator \( B_\alpha \) is continuous from \( L^2((0, T) \times (\omega_1, \omega_2)) \) to \( L^2((0, T) \times (\omega_1, \hat{\omega}_2)) \), i.e., there exists a constant \( C > 0 \) such that

\[
C \int_{\omega_1}^{\hat{\omega}_2} \int_0^T |\phi(t, x)|^2 dt dx \geq \int_{\omega_1}^{\hat{\omega}_2} \int_0^T |B_\alpha \phi(t, x)|^2 dt dx,
\]

for every function \( \phi \) for which both integrals are defined.

Proof: Let us observe that

\[
B_\alpha \phi(t, x) = \sum_{(p, q) \in S} \phi(t + pa(x - \omega_1), x + qa(x - \omega_1)),
\]

where

\[
S = \{(2, 1), (1, 2), (0, 1), (1, 0)\}.
\]

Then,

\[
\int_{\omega_1}^{\hat{\omega}_2} \int_0^T |B_\alpha \phi(t, x)|^2 dt dx \leq 4 \sum_{(p, q) \in S} \int_{\omega_1}^{\hat{\omega}_2} \int_0^T |\phi(t + pa(x - \omega_1), x + qa(x - \omega_1))|^2 dt dx.
\]

To estimate the integral

\[
\int_{\omega_1}^{\hat{\omega}_2} \int_0^T |\phi(t + pa(x - \omega_1), x + qa(x - \omega_1))|^2 dt dx
\]

we make the change of variables

\[
\xi = t + pa(x - \omega_1), \quad \eta = x + qa(x - \omega_1).
\]

In these new variables, (37) is written as

\[
(1 + qa) \int \int_{\Omega_{p, q}} |\phi(\xi, \eta)|^2 d\xi d\eta,
\]

where \( \Omega_{p, q} \) is the image of \( (0, \hat{T}) \times (\omega_1, \hat{\omega}_2) \) under the map defined by (38). Besides, in view of (35), for any \((p, q) \in S, \Omega_{p, q} \subset (0, T) \times (\omega_1, \omega_2)\).

Consequently,

\[
\int_{\omega_1}^{\hat{\omega}_2} \int_0^T |\phi(\xi, \eta)|^2 d\xi d\eta \geq \int \int_{\Omega} |\phi(\xi, \eta)|^2 d\xi d\eta.
\]

This fact, in account of the inequality (36), proves the proposition.

Proposition 5: For every \( \rho, \ell \in \mathbb{R} \) and \( n \in \mathbb{N} \) the following equality holds

\[
B_\alpha \left( e^{\frac{in\pi t}{\ell}} \sin \frac{n\pi x}{\ell} \right) = 4e^{\frac{in\pi t}{\ell} (t + x - \omega_1)} \sin \frac{n\pi x}{\ell} \sin \left( \frac{n\pi(x - \omega_1)}{2\ell} \alpha \right) \sin \left( \frac{n\pi(x - \omega_1)}{2\ell} \beta \right),
\]

where \( \alpha = (\rho^{-1} + a) \) and \( \beta = (\rho^{-1} - a) \).

Remark 2: If \( \phi(t, x) \) is a solution of the wave equation

\[
\rho^2 \phi_{tt} - \phi_{xx} = 0, \quad \phi(t, 0) = \phi(t, \ell) = 0,
\]

whose initial data \( \phi|_{t=0} \) and \( \phi_t|_{t=0} \) are finite linear combinations of the eigenfunctions \( \left( \sin \frac{n\pi x}{\ell} \right) \) then

\[
B_{\rho^{-1}} \phi(t, x) = 0.
\]
Proposition 6: Let \( \ell, \alpha \neq \beta \) be positive numbers and \( I \) an interval in \( \mathbb{R} \). Then there exists a constant \( C > 0 \) such that, for every \( n \in \mathbb{R} \),

\[
\int_I \left| \sin \frac{n\pi x}{\ell} \sin \left( \frac{n\pi(x - \omega_1)}{2\ell} \right) \sin \left( \frac{n\pi\omega_1}{2\ell} \right) \right|^2 dx \geq C.
\]

This fact may be easily proved by computing the integral.

Proof: [Proof of Theorem 2] Let \( (a_n)_{n \in \mathbb{Z}}, (\alpha_n)_{n \in \mathbb{Z}} \) be complex finite sequence satisfying \( a_n = \overline{a_n} \) and 

\[
\phi^k(t, x) = \sum_{n \in \mathbb{Z}} a_n^k e^{\frac{in\pi}{\ell}x}, \quad k = 1, 2.
\]

We take \( a = \rho_1^{-1} \). Since \( T > T_0 \), it is possible to choose \( \hat{\omega}_2 > \omega_1 \) sufficiently close to \( \omega_1 \) in such a way that \( \hat{\omega}_2 \) and \( \hat{T} \) satisfy (35) and in addition \( \hat{T} \geq T_0 \).

Then, according to Proposition 4,

\[
C \int_{\omega_1}^{\omega_2} \int_0^T |\phi^1 + \phi^2|^2 dtdx \geq \int_{\omega_1}^{\omega_2} \int_0^T |B_a \phi^1 + B_a \phi^2|^2 dtdx. \quad (39)
\]

But, in view of Remark 2,

\[
B_a \phi^1 = 0.
\]

Thus, from the inequality (39) it holds

\[
C \int_{\omega_1}^{\omega_2} \int_0^T |\phi^1 + \phi^2|^2 dtdx \geq \int_{\omega_1}^{\omega_2} \int_0^{\hat{T}} |B_a \phi^2|^2 dtdx. \quad (40)
\]

On the other hand, since

\[
\phi^2(t, x) = \sum_{n \in \mathbb{Z}} a_n^2 e^{\frac{in\pi}{\ell}x} \sin \frac{n\pi x}{\ell},
\]

Proposition 5 guarantees that

\[
B_a \phi^2 (t, x) = \sum_{n \in \mathbb{Z}} a_n^2 e^{\frac{in\pi}{\ell}x} \Theta_n(x),
\]

where

\[
\Theta_n(x) := 4e^{\frac{in\pi}{\ell}x} \sin \frac{n\pi x}{\ell} \sin \left( \frac{n\pi(x - \omega_1)}{2\ell} \right) \sin \left( \frac{n\pi\omega_1}{2\ell} \right).
\]

with

\[
\alpha = \frac{1}{\rho_2} + \frac{1}{\rho_1}, \quad \beta = \frac{1}{\rho_2} - \frac{1}{\rho_1}.
\]

Besides, in view of Proposition 6, there exists a constant \( C > 0 \) such that, for every \( n \in \mathbb{N} \) the following inequality is verified

\[
\int_{\omega_1}^{\omega_2} |\Theta_n(x)|^2 dx \geq C. \quad (41)
\]

Then, since \( \hat{T} \geq T_0 \geq 2\rho_2 \ell_2 \),

\[
\int_{\omega_1}^{\omega_2} \int_0^{\hat{T}} |B_a \phi^2|^2 dtdx \geq \int_{\omega_1}^{\omega_2} \int_0^{2\rho_2 \ell_2} \left| \sum_{n \in \mathbb{Z}} a_n^2 e^{\frac{in\pi}{\ell}x} \Theta_n(x) \right|^2 dtdx = 2 \sum_{n \in \mathbb{N}} |a_n^2|^2 \int_{\omega_1}^{\omega_2} |\Theta_n(x)|^2 dx,
\]

(we have used the fact that the functions \( e^{\frac{in\pi}{\ell}x} \) are orthogonal in \( (0, 2\rho_2 \ell_2) \)) and consequently, in view of (40) and (41),

\[
C \int_{\omega_1}^{\omega_2} \int_0^T |\phi^1 + \phi^2|^2 dtdx \geq \sum_{n \in \mathbb{N}} |a_n^2|^2.
\]
The inequality
\[ C \int_{\omega_1}^{\omega_2} \int_0^T |\phi^1 + \phi^2|^2 dt \, dx \geq \sum_{n \in \mathbb{N}} |a_n|^2 \]
is proved in a similar way by applying to \(\phi^1 + \phi^2\) the operator \(B_a\) con \(a = \rho_2^{-1}\). This concludes the proof of the theorem.

3) Control of strings with equal densities: Theorem 2 does not provide any information on what happens when \(\rho_1 = \rho_2\) but \(\ell_1 \neq \ell_2\). This fact is due to the local character of the operators \(B_a\), which does not allow them to distinguish between waves that propagate with the same speed. This fact, however, is not a purely technical one: if \(\rho_1 = \rho_2 = \rho\) it is not sufficient that \(\ell_1 \neq \ell_2\) for the inequality (34) to hold.

In fact, let us assume that
\[ \frac{\ell_1}{\ell_2} = \frac{p}{q}, \quad p, q \in \mathbb{N}. \]
Then, the solutions
\[ \phi^1(t, x) = e^{\frac{ip}{q}lx} \sin \frac{p\pi x}{\ell_1}, \quad \phi^2(t, x) = -e^{\frac{ip}{q}lx} \sin \frac{q\pi x}{\ell_2} \]
satisfy
\[ \phi^1(t, x) + \phi^2(t, x) \equiv 0, \]
and then, an inequality of the type (34) is impossible for any interval \(\omega\) and any time \(T\) whenever the ratio \(\frac{\ell_1}{\ell_2}\) is an irrational number. In this sense the problem turns out to be similar to the simultaneous controllability problem of two strings from one end. Using the same technique as in the proof of Theorem 1, it is possible to prove an inequality weaker than (34), which implies the controllability of some subspace of initial states characterized in terms of their Fourier coefficients.

Let us denote by \(Z^k, k = 1, 2,\) the space of all the finite linear combinations of the functions \((\sin \frac{n\pi x}{\ell_k})_{n \in \mathbb{N}}\).

**Theorem 3:** Let \(\rho_1 = \rho_2 = \rho\) and \(T \geq 2p(\ell_1 + \ell_2)\). There exists a constant \(C > 0\) such that
\[ C \int_{\omega}^{\omega_2} \int_0^T |\phi^1(t, x) + \phi^2(t, x)|^2 dt \, dx \geq \sum_{n \in \mathbb{N}} \sin^2 \frac{\ell_2 n\pi}{\ell_1} \left((\phi_n)^2 + n^{-2}(\phi_1)^2\right), \quad (42) \]
\[ C \int_{\omega}^{\omega_2} \int_0^T |\phi^1(t, x) + \phi^2(t, x)|^2 dt \, dx \geq \sum_{n \in \mathbb{N}} \sin^2 \frac{\ell_1 n\pi}{\ell_2} \left((\phi_n)^2 + n^{-2}(\phi_1)^2\right), \quad (43) \]
for all the solutions \(\phi^1, \phi^2\) of (32) with initial states in \(Z^1 \times Z^1\) and \(Z^2 \times Z^2\), respectively.

**Proof:** The inequality (43) is equivalent to
\[ C \int_{\omega}^{\omega_2} \int_0^T |\Phi|^2 dt \, dx \geq \sum_{n \in \mathbb{N}} |a_n|^2 \sin^2 \frac{\ell_1 n\pi}{\ell_2}, \]
where
\[ \Phi := \sum_{n \in \mathbb{Z}} a_n e^{\frac{in\pi x}{\ell_1}} \sin \frac{n\pi x}{\ell_1} + a_n e^{\frac{in\pi x}{\ell_2}} \sin \frac{n\pi x}{\ell_2}, \]
for all the complex finite sequences verifying \(a_{-n} = \overline{a_n}, a_{-n} = \overline{a_n}.

In order to prove this assertion, let us observe that, for every \(x \in \omega,\)
\[ (\rho \ell_1)^{-1} \left( \sum_{n \in \mathbb{Z}} a_n e^{\frac{in\pi x}{\ell_1}} \right) = 0, \quad (44) \]
where \((\rho \ell_1)^{-1}\) is the operator defined by (10) for the number \(\rho \ell_1\) (the equality (44) corresponds to the \(2\rho \ell_1\)-periodicity of the solution \(\phi^1\)). Then,
\[ (\rho \ell_1)^{-1} \Phi = (\rho \ell_1)^{-1} \sum_{n \in \mathbb{Z}} a_n e^{\frac{in\pi x}{\ell_2}} \sin \frac{n\pi x}{\ell_2} \]
\[ = \sum_{n \in \mathbb{Z}} a_n e^{\frac{in\pi x}{\ell_2}} \sin \frac{\ell_1 n\pi}{\ell_2} \sin \frac{n\pi x}{\ell_2}. \]
Besides, in view of the continuity of the operator \((\rho_{\ell_1})^-\) we obtain that, for every \(x \in \omega\),
\[
\int_0^T |\Phi|^2 \, dt \geq \int_{\rho_{\ell_1}}^{T-\rho_{\ell_1}} |(\rho_{\ell_1})^-\Phi|^2 \, dt = \int_{\rho_{\ell_1}}^{T-\rho_{\ell_1}} \left| \sum_{n \in \mathbb{Z}} a_n^2 e^{in\pi x} \sin \frac{n\pi x}{\ell_2} \sin \frac{n\pi x}{\ell_2} \right|^2 \, dt.
\] (45)

On the other hand, since \(T \geq 2\rho(\ell_1 + \ell_2)\),
\[
\int_{\rho_{\ell_1}}^{T-\rho_{\ell_1}} \sum_{n \in \mathbb{Z}} a_n^2 e^{in\pi x} \sin \frac{\ell_1 n\pi}{\ell_2} \sin \frac{n\pi x}{\ell_2} \, dt \geq \int_{\rho_{\ell_1}}^{\rho_{\ell_1} + 2\rho_{\ell_2}} \sum_{n \in \mathbb{Z}} a_n^2 e^{in\pi x} \sin \frac{\ell_1 n\pi}{\ell_2} \sin \frac{n\pi x}{\ell_2} \, dt
\]
\[
= 2 \sum_{n \in \mathbb{N}} |a_n^2|^2 \sin^2 \frac{n\pi x}{\ell_2} \sin^2 \frac{\ell_1 n\pi}{\ell_2}.
\]

(we have used the fact that the functions \(\left( e^{in\pi x} \right)_{n \in \mathbb{Z}} \) are orthogonal in any interval of length \(2\rho_{\ell_2}\)).

Further, in view of (45),
\[
C \int_0^T \int_0^T |\Phi|^2 \, dtdx \geq \sum_{n \in \mathbb{N}} |a_n^2|^2 \sin^2 \frac{\ell_1 n\pi}{\ell_2} \int_\omega \sin^2 \frac{n\pi x}{\ell_2} \, dx.
\] (46)

Finally, let us observe that for any interval \(\omega \subset \mathbb{R}\) there exists a constant \(C = C(\omega)\) such that
\[
\int_\omega \sin^2 \frac{n\pi x}{\ell_2} \, dx \geq C.
\]

Thus, from (46) it holds
\[
C \int_0^T \int_0^T |\Phi|^2 \, dtdx \geq \sum_{n \in \mathbb{N}} |a_n^2|^2 \sin^2 \frac{\ell_1 n\pi}{\ell_2}.
\]

The inequality (42) is proved in an analogous way.

**Corollary 5:** If \(\ell_1 / \ell_2\) is an irrational number and \(T \geq 2\rho(\ell_1 + \ell_2)\) then the system (30) is spectrally controllable in time \(T\), i.e., all the initial states \((u_0^1, u_1^1) \in Z^1 \times Z^1\), \((u_0^2, u_1^2) \in Z^2 \times Z^2\) are controllable in time \(T\).

If additional information on the ratio \(\ell_1 / \ell_2\) is provided, then it is possible to describe subspaces of controllable initial states in a way completely analogous as it was done in section 1.

**Corollary 6:** a) If \(\ell_1 / \ell_2 \in B_\varepsilon\) then, the subspace of initial states
\[
(u_0^i, u_1^i) \in \tilde{H}^{2+\varepsilon}(0, \ell_1) \times \tilde{H}^{1+\varepsilon}(0, \ell_1),
\]
is controllable in any time \(T \geq 2\rho(\ell_1 + \ell_2)\). In particular, if \(\ell_1 / \ell_2\) is an algebraic irrational number, this subspace is controllable for any \(\varepsilon > 0\).

b) If \(\ell_1 / \ell_2\) admits a bounded representation as a continuous fraction then, the subspace of initial states
\[
(u_0^i, u_1^i) \in \left[ H^2(0, \ell_1) \cap H_0^1(0, \ell_1) \right] \times H_0^1(0, \ell_1),
\]
is controllable in any time \(T \geq 2\rho(\ell_1 + \ell_2)\).

**B. Simultaneous control on the whole domain**

Let \(\Omega\) a bounded open set in \(\mathbb{R}^n\) with smooth boundary and \(f \in L^2_{\text{loc}}(\mathbb{R}^{n+1})\). We consider the system
\[
\begin{cases}
\rho_{\ell_1}^2 u_{\ell_1}^k - \Delta u^k + f = 0 & \text{in } \Omega \times \mathbb{R}, \\
u^k |_{\partial \Omega} = 0 & \text{in } \mathbb{R}, \\
u^k (0,.) = u_0^k, \quad u_0^k (0,.) = u_0^k \text{in } \Omega.
\end{cases}
\] (47)

The system (47) describes the motion of \(N\) elastic membranes with densities \(\rho_1, \ldots, \rho_N\) having at rest the same shape \(\Omega\), whose borders are fixed. These membranes are controlled through a function \(f\), which acts on the whole domain \(\Omega\). When \(n = 1\) the system (47) is a particular case of the system (30) with \(\ell_1 = \ell_2\) and \(\omega = (0, \ell_1)\).
The problem (47) is well posed for initial states \((u_0^k, u_T^k) \in H_0^1(\Omega) \times L^2(\Omega), k = 1, \ldots, N\). When \(f = 0\) we obtain the homogeneous system

\[
\begin{align*}
\rho_k^2 \phi_{tt}^k - \Delta \phi^k &= 0 & \quad & \text{in } \Omega \times \mathbb{R}, \\
\phi^k |_{\partial \Omega} &= 0 & \quad & \text{in } \mathbb{R}, \\
\phi^k(0, \cdot) &= \phi_0^k, & \phi_T^k(0, \cdot) &= \phi_T^k \text{ on } \Omega,
\end{align*}
\]

(48)

which is also well posed for initial states \((\phi_0^k, \phi_T^k) \in L^2(\Omega) \times H^{-1}(\Omega), k = 1, \ldots, N\).

If \((\mu_n)_{n \in \mathbb{N}}\) is the increasing sequence of the eigenvalues con the Laplace operator \(-\Delta\) with homogeneous Dirichlet boundary conditions in \(\Omega\) and \((\theta_n)_{n \in \mathbb{N}}\) is orthonormal basis of \(L^2(\Omega)\) formed by the corresponding eigenfunctions, then the solutions of (48) are determined by the formulas

\[
\phi^k(t, x) = \sum_{n \in \mathbb{Z}} a_n^k e^{\frac{i \lambda_n t}{\mu_n}} \theta_{|n|}(x)
\]

where

\[
\lambda_n = \sqrt{\mu_{|n|}} \\text{sgn } n, \quad n \in \mathbb{Z}.
\]

\[
a_n^k = \frac{1}{2} \left( \phi_{0,n}^k + i \lambda_n \phi_{1,n}^k \right), \quad k = 1, \ldots, N, \quad n \in \mathbb{Z}.
\]

The control problem associated to the system (47) is: *Given \(T > 0\), to determine for which initial states \((u_0^k, u_T^k) \in H_0^1(\Omega) \times L^2(\Omega), k = 1, \ldots, N\), there exists \(f \in L^2(\Omega)\) such that the solution of (47) satisfies*

\[
u_T^k |_{t = T} = u_T^k |_{t = T} = 0.
\]

When this is possible for all the initial states from \(H_0^1(\Omega) \times L^2(\Omega)\) we say that (47) is *exactly controllable in time T*.

The application of HUM guarantees that the system (47) is exactly controllable in time \(T\) if, and only if, there exists a constant \(C > 0\) such that the inequality

\[
C \int_0^T \int_\Omega \left| \sum_{k=1}^N \phi^k(t, x) \right|^2 \, dt \, dx \geq \sum_{k=1}^N \left| (\phi_0^k, \phi_T^k) \right|^2_{L^2(\Omega) \times H^{-1}(\Omega)},
\]

(49)

holds for all the solutions of (48) with initial states \((\phi_0^k, \phi_T^k) \in L^2(\Omega) \times H^{-1}(\Omega), k = 1, \ldots, N\).

This fact is equivalent to the existence of a constant \(C > 0\) such that

\[
C \int_0^T \int_\Omega \left| \sum_{k=1}^N \sum_{n \in \mathbb{Z}} a_n^k e^{\frac{i \lambda_n t}{\mu_n}} \theta_{|n|}(x) \right|^2 \, dt \, dx \geq 2 \sum_{k=1}^N \sum_{n \in \mathbb{Z}} \left| a_n^k \right|^2,
\]

(50)

for all the complex finite sequences \((a_n^k)_{n \in \mathbb{Z}}, k = 1, \ldots, N\), verifying \(a_{-n}^k = \overline{a_n^k}\).

**Theorem 4**: The system (47) is exactly controllable in time \(T > 0\) if, and only if, the numbers \(\rho_1, \ldots, \rho_N\) are all different.

*Proof*: If two of the numbers \(\rho_1, \ldots, \rho_N\) coincide, say \(\rho_1 = \rho_2\), then if we choose

\[
a_1^n = -a_2^n, \quad a_n^0 = 0, \quad k > N,
\]

the inequality (50) becomes

\[
0 \geq 4 \sum_{n \in \mathbb{N}} \left| a_1^n \right|^2,
\]

what is not true in general. Thus, if two of the numbers \(\rho_1, \ldots, \rho_N\) coincide, there exist finite sequences for which (50) is false.
Let us observe now that, due to the orthogonality in $L^2(\Omega)$ of the functions $(\theta_n)_{n \in \mathbb{N}}$, the inequality (50) may be written as
\[
C \int_0^T \sum_{n \in \mathbb{N}} \int_0^T \left| \sum_{k=1}^N a_n^k e^{\frac{i \lambda_n t}{\rho_k}} \right|^2 dt \geq \sum_{n \in \mathbb{N}} \sum_{k=1}^N |a_n^k|^2.
\]
Consequently, for any $T > 0$ and distinct numbers $\rho_1, ..., \rho_N$ it suffices to apply Proposition 7, which is proved in the sequel, to obtain the inequality (50) and so, the proof of the theorem.

**Proposition 7:** Let $\rho_1, ..., \rho_N$ be distinct positive numbers and $(\lambda_n)_{n \in \mathbb{N}}$ a sequence of real positive numbers that tends to infinity. Then, for all $T > 0$ there exists a constant $C = C(T, N, \rho_1, ..., \rho_N) > 0$ such that
\[
\int_0^T \left| \sum_{k=1}^N a^k e^{\frac{i \lambda_n t}{\rho_k}} \right|^2 dt \geq C \sum_{k=1}^N |a^k|^2,
\]
for any $n \in \mathbb{N}$ and $(a^1, ..., a^N) \in \mathbb{R}^N$.

**Proof:** We proceed by induction with respect to the number $N$. For $N = 1$ the inequality is immediate. Let us suppose that the inequality is true for $N - 1$.

Let us denote
\[
I_n = \sum_{k=2}^N a^k e^{\frac{i \lambda_n t}{\rho_k}}.
\]
Then, according to the induction hypothesis, there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$,
\[
\int_0^T |I_n|^2 dt \geq C \sum_{k=2}^N |a^k|^2.
\]
On the other hand
\[
\int_0^T \left| \sum_{k=1}^N a^k e^{\frac{i \lambda_n t}{\rho_k}} \right|^2 dt = |a^1|^2 T + 2 \Re(\int_0^T a^1 e^{\frac{i \lambda_n t}{\rho_1}} T_n dt) + \int_0^T |I_n|^2 dt.
\]
(51)

Let us observe that
\[
\left| \Re \int_0^T a^1 e^{\frac{i \lambda_n t}{\rho_1}} T_n dt \right| \leq |a^1| \left| \int_0^T e^{\frac{i \lambda_n t}{\rho_1}} T_n dt \right| = |a^1| \left| \sum_{k=2}^N a^k \gamma_{n,k} dt \right|,
\]
where
\[
\gamma_{n,k} = \int_0^T e^{\frac{1}{\rho_k} - \frac{1}{\rho_1}} i \lambda_n t dt.
\]
(52)
Besides,
\[
|a^1| \left| \sum_{k=2}^N a^k \gamma_{n,k} \right| \leq |a^1| \sum_{k=2}^N |a^k| |\gamma_{n,k}| \leq \frac{1}{2} \sum_{k=2}^N \left( |a^1|^2 + |a^k|^2 \right) |\gamma_{n,k}|.
\]
(53)
Finally, from (52) and (53) it follows the inequality
\[
\left| \Re \int_0^T a^1 e^{\frac{i \lambda_n t}{\rho_1}} T_n dt \right| \leq \frac{1}{2} \sum_{k=2}^N \left( |a^1|^2 + |a^k|^2 \right) |\gamma_{n,k}|.
\]
This latter inequality, in view of (51), implies
\[
\int_0^T \left| \sum_{k=1}^N a^k e^{\frac{i \lambda_n t}{\rho_k}} \right|^2 dt \geq |a^1|^2 (T - \sum_{k=2}^N |\gamma_{n,k}|) + \sum_{k=2}^N |a^k|^2 (C - |\gamma_{n,k}|).
\]
(54)
Let us observe now that, for every $k = 2, \ldots, N$,

$$|\gamma_{n,k}| \leq \frac{2|\rho_k - \rho_1|}{\rho_1 \rho_k \lambda_n} \rightarrow 0.$$ 

Therefore, there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$,

$$\sum_{k=2}^{N} |\gamma_{n,k}| \leq \frac{T}{2}, \quad |\gamma_{n,k}| \leq \frac{C}{2}, \quad k = 2, \ldots, N.$$ 

Consequently, from (54) it holds that, for every $n \geq n_0$,

$$\int_{0}^{T} \left| \sum_{k=1}^{N} a^k e^{\frac{i \lambda n t}{\rho_k}} \right|^2 dt \geq |a|^2 \frac{T}{2} + \frac{C}{2} \sum_{k=2}^{N} |a^k|^2 \geq C \sum_{k=1}^{N} |a^k|^2.$$  \hspace{1cm} (55)

Finally, it suffices to note that the functions $e^{\frac{i \lambda n t}{\rho_k}}$, $k = 1, \ldots, N$, are linearly independent over any interval, and thus,

$$\int_{0}^{T} \left| \sum_{k=1}^{N} a^k e^{\frac{i \lambda n t}{\rho_k}} \right|^2 dt > 0,$$

except when $a^1 = \ldots = a^N = 0$. This allows to apply a standard compactness-uniqueness argument to prove that, for every $n \in \mathbb{N}$, there exists a constant $C_n > 0$ such that

$$\int_{0}^{T} \left| \sum_{k=1}^{N} a^k e^{\frac{i \lambda n t}{\rho_k}} \right|^2 dt \geq C_n \sum_{k=1}^{N} |a^k|^2.$$ 

Therefore, there exists a constant $C > 0$ such that

$$\int_{0}^{T} \left| \sum_{k=1}^{N} a^k e^{\frac{i \lambda n t}{\rho_k}} \right|^2 dt \geq C \sum_{k=1}^{N} |a^k|^2,$$

for any $n < n_0$. This fact, in view of (55), provides the assertion of the proposition. \hspace{1cm} \blacksquare

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It is precisely in this point where we use in an essential way the fact that the numbers $\rho_k$, $k = 1, \ldots, N$, are all distinct.