The Linearized Benjamin-Bona-Mahony Equation: A Spectral Approach to Unique Continuation

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Abstract

This paper is devoted to analyzing a unique continuation problem for the linearized Benjamin-Bona-Mahony equation with space-dependent potentials in a bounded interval with Dirichlet boundary conditions. The underlying Cauchy problem is a characteristic one. We prove two unique continuation results by means of spectral analysis and the (generalized) eigenvector expansion of the solution, instead of the usual Holmgren-type method or Carleman-type estimates. It is found that the unique continuation property depends very strongly on the nature of the potential and, in particular, on its zero set, and not only on its boundedness or integrability properties.

2000 Mathematics Subject Classification. Primary 35B60; Secondary 47A70, 47B07.

1 Introduction

This paper is devoted to analyze some basic aspects of a linearized version of the well known Benjamin-Bona-Mahony equation

\[
\begin{cases}
  u_t - u_{txx} + u_x + uu_x = 0 & \text{in } (0, T) \times (0, 1), \\
  u(t, 0) = u(t, 1) = 0 & \text{in } (0, T), \\
  u(0, x) = u_0(x) & \text{in } (0, 1).
\end{cases}
\]

System (1.1) is a model for long waves in nonlinear dispersive systems, which is an alternative model of the classical KdV equation.

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The system we shall consider reads as follows:

\[
\begin{align*}
\begin{cases}
  u_t - u_{txx} &= [\alpha(x)u]_x + \beta(x)u & \text{in } (0,T) \times (0,1), \\
  u(t,0) &= u(t,1) = 0 & t \in (0,T), \\
  u(0,x) &= u_0(x) & \text{in } (0,1),
\end{cases}
\end{align*}
\] 

(1.2)

where \( T > 0 \) is a given time, \( \alpha(\cdot) \in L^\infty(0,1) \) and \( \beta(\cdot) \in L^2(0,1) \) are given potentials.

System (1.2) is indeed a linearized version of (1.1). It is however worth mentioning that in (1.2) we have taken potentials that do not depend in time, which corresponds to a linearization around a time-independent solution. The only time-independent solution of (1.1) is the trivially zero one, in which case \( \alpha \equiv -1 \) and \( \beta \equiv 0 \). However, the analysis of (1.2) is of independent interest since it covers the linearization of a larger range of equations like (1.1).

It is easy to show that for any \( u_0 \in L^2(0,1) \), system (1.2) admits a unique weak solution \( u \in C([0,T];L^2(0,1)) \).

Let \( F \subset (0,1) \) be an open, non-empty subset. We are interested in the property of unique continuation for (1.2), that is, whether

\[
u(t,x) = 0 \quad \text{in } (0,T) \times F
\]

(1.3)

implies that \( u \) vanishes identically.

This problem is motivated by questions related to the controllability and the stabilization of Benjamin-Bona-Mahony like equations. Indeed, if one adds a locally distributed damping in system (1.1) like

\[
u_t - u_{txx} + u_x + uu_x = -\chi_F u,
\]

then the energy of solutions is dissipated. Here and in the sequel \( \chi_F \) denotes the characteristic function of the set \( F \). More precisely, let \( \mathcal{E} \) be the energy of solutions:

\[
\mathcal{E}(t) = \int_0^1 [u^2 + u_x^2]dx.
\]

By means of the classical energy method, using the boundary conditions in (1.1), we get,

\[
\frac{d\mathcal{E}(t)}{dt} = -2 \int_F u^2(x,t)dx.
\]

It is then natural to ask whether, as \( t \to \infty \), solutions tend to zero. When doing this, one may apply La Salle’s invariance principle (in strong or weak topologies depending on the compactness properties of trajectories) but nothing can be said if one does not have a positive answer to the unique continuation property above that asserts that the only non dissipated trajectory is the trivial one. This programme was succesfully developed in [9] in the context of the KdV equations. However, the BBM-like models under consideration present important new features that need further analysis. The first one is the unique continuation problem formulated above.
It is important to say that the techniques we discuss along this paper, based on spectral analysis, certainly do not suffice to address the general problem when coefficients depend both in space and time.

There exists a very extensive literature on unique continuation problems. Moreover, the equation under consideration is $1 - D$ in space and therefore, one could expect the problem above to fit in some of the existing results. But this does not seem to be the case.

First of all, we note that the Benjamin-Bona-Mahony equation although it is a Partial Differential Equation, it looks rather as an anti-PDE providing time analyticity but not gain of regularity in space. This is related to the fact that (1.1) may be written as

$$u_t + (I - \partial_x^2)^{-1}[u_x + uu_x] = 0.$$  

Thus, we see that the generator of the underlying semigroup is given by $(I - \partial_x^2)^{-1}\partial_x$ which is a compact operator, in opposition with most frequent situation in PDE where the generator is an unbounded operator.

This special nature of the system under consideration determines also a strange behavior of the characteristic lines. Indeed, both $x = \text{Const.}$ and $t = \text{Const.}$ are characteristic lines for the first equation in (1.2). Hence the uniqueness or unique continuation problem we are discussing is a characteristic Cauchy problem. Therefore we can not apply Holmgren uniqueness theorem even in the simplest case in which $\alpha(\cdot)$ and $\beta(\cdot)$ are analytic functions.

On the other hand, when coefficients fail to be analytic, the main tools to prove unique continuation properties are the so-called Carleman-type estimates ([7], [12], for instance). In Carleman-type estimates the lower order terms (with bounded coefficients or even with unbounded coefficients under suitable integrability conditions) of the equation can be controlled in some weighted norms by the principal part of the operator.

However, for our problem, the unique continuation property depends very strongly on qualitative properties of the coefficients of lower order terms, $\alpha(\cdot)$ and $\beta(\cdot)$, and not only on its size! In fact, we have the following simple negative result:

**Example 1.1** Let $\alpha(\cdot) = \beta(\cdot) \equiv 0$. Then any time-independent function $u = u(x) \in C_0^\infty((0,1) \setminus \mathcal{F})$ satisfies (1.2) and (1.3). Therefore, the unique continuation property does not hold for this simple case unless $(0,1) \setminus \mathcal{F} = \emptyset$.

Therefore, in the present situation, one can not expect to apply Carleman-type estimates either that are typically of application for non-characteristic Cauchy problems.

Note also that, according to this counterexample, the unique continuation result stated in [3] does not hold without further additional assumptions.

In summary, due to the very particular structure of the equation under consideration, the “usual” Holmgren and Carleman methods do not apply to our problem. Thus, in order to solve our unique continuation problem, we develop a spectral method which is reminiscent of the classical and historical method of Fourier of developing solutions on the basis of complex exponentials or trigonometric polynomials.
To conclude this introduction we mention the work by S. Micu [8], who addressed the very closely connected question of unique continuation in the particular case in which \( \alpha \) is constant and the potential \( \beta \) vanishes. There it was proved that the only solution such that \( u_x \) vanishes on the boundary is the trivial one. We note however that the fact of considering variable coefficients requires of important further developments on the spectral analysis.

The proofs of the results presented in this paper are given in [11].

The rest of this paper is organized as follows. In section 2 we present the main unique continuation results and also some applications to the stabilization problem. Sections 3 and 4 are devoted to present a sketch of the proof of the main unique continuation results. We close in section 5 with some further comments and the formulation of some open problems.

2 Main Results

First, we have the following necessary condition for unique continuation of (1.2)–(1.3):

**Theorem 2.1** Let \( F \subset (0, 1) \) be an open, non-empty subset, \( \alpha(\cdot) \in L^\infty(0, 1) \) and \( \beta(\cdot) \in L^2(0, 1) \). Suppose that the unique continuation property above holds. Then \( \alpha(\cdot) \) and \( \beta(\cdot) \) can not vanish simultaneously in any open, non-empty subset of \((0, 1) \setminus F\).

In view of Theorem 2.1, we need to impose conditions on potentials \( \alpha \) and \( \beta \), and more precisely in their zero sets in order for the unique continuation property to be true. Therefore, it is natural to analyze under what conditions on \( \alpha \) and \( \beta \), the unique continuation for (1.2) and (1.3) holds.

For this purpose, we need to introduce some notations. We denote by \( W(0, 1) \) the set of all weight functions on \((0, 1)\), i.e. the set of all measurable, bounded functions which are positive almost everywhere in \((0, 1)\). For any \( \alpha(\cdot) \in W(0, 1) \), we denote by \( L^2(0, 1; \alpha) \) the Hilbert space of the completion of \( C_0^\infty(0, 1) \) with respect to the norm

\[
\|f\|_{2,\alpha} \triangleq \int_0^1 \alpha(x)|f(x)|^2\,dx, \quad \forall f \in C_0^\infty(0, 1).
\]

(2.1)

We have the following unique continuation result.

**Theorem 2.2** Let \( \beta(\cdot) = 0 \) and either \( \alpha(\cdot) \in W(0, 1) \) or \( -\alpha(\cdot) \in W(0, 1) \). Let \( 0 < a < b < 1 \), \( T > 0 \) and \( u_0 \in L^2(0, 1) \). Suppose that the weak solution \( u \in C([0, T]; L^2(0, 1)) \) of (1.2) satisfies (1.3) with \( F = (0, a) \cup (b, 1) \). Then \( u \equiv 0 \) in \( \mathbb{R} \times (0, 1) \).

Note that in Theorem 2.2 it is assumed that \( u \) vanishes in a neighborhood of both extremes \( x = 0, 1 \) of the interval \((0, 1)\) and that \( \beta \equiv 0 \). If we impose more regularity conditions on \( \alpha \), we have the following better result, which allows, in particular, a non-zero potential \( \beta \).

**Theorem 2.3** Let \( \beta(\cdot) \in L^\infty(0, 1) \) and \( \alpha(\cdot) \in W^{2,\infty}(0, 1) \) with \( \min_{x \in [0,1]} |\alpha(x)| > 0 \). Let \( 0 \leq a < b \leq 1 \), \( T > 0 \) and \( u_0 \in L^2(0, 1) \). Suppose that the weak solution \( u \in C([0, T]; L^2(0, 1)) \) of (1.2) satisfies (1.3) with \( F = (a, b) \). Then \( u \equiv 0 \) in \( \mathbb{R} \times (0, 1) \).
Remark 2.1 Theorems 2.1–2.3 show that the unique continuation property for system (1.2) under condition (1.3) depends very strongly in the nature of the coefficients $\alpha(\cdot)$ and $\beta(\cdot)$ of the lower order terms of the first equation in (1.2). As far as we know, such a phenomenon was not observed and analyzed in the existing literature.

Remark 2.2 Taking $\alpha(x) \equiv -1$ and $\beta(x) \equiv 0$ in Theorem 2.3, we obtain the unique continuation result in [8].

To end this section, we give some applications of Theorems 2.2 and 2.3. Let us consider the stabilization problem of the following Benjamin-Bona-Mahony like equation:

\[
\begin{cases}
  u_t - u_{txx} = [\alpha(x)u]' - \chi_F(x)u & \text{in } (0, \infty) \times (0, 1), \\
  u(t, 0) = u(t, 1) = 0 & t \in (0, \infty), \\
  u(0) = u_0 & \text{in } (0, 1),
\end{cases}
\]  

(2.2)

We denote the energy of solution of (2.2) by

\[ E(u(t)) \triangleq \int_0^1 [u^2(t, x) + u_x^2(t, x)]dx. \]

We have the following result.

**Theorem 2.4** Let $\alpha \in W(0, 1) \cap H^1(0, 1)$, $1/\alpha \in L^1(0, 1)$ and $\alpha'(x) \leq 0$ a.e. $(0, 1)$. (2.3)

Let $0 < a < b < 1$, and $F = (0, a) \cup (b, 1)$. Then for any $u_0 \in H_0^1(0, 1)$, $u(t)$ tends to 0 in $H_0^1(0, 1)$ weakly as $t \to \infty$.

Furthermore, if $u_0 \in U_2$, where $U_2$ is the subspace in $H_0^1(0, 1)$ spanned by the following space

\[ V_2 \triangleq \left\{ v_0 \in H_0^1(0, 1) \cap H^2(0, 1); \int_0^1 e^{-\int_0^s \frac{\chi_F(x)}{\alpha(x)}dx} [v_0(x) - v_{0,xx}(x)]dx = 0 \right\}, \]

then $E(u(t))$ tends to 0 as $t \to \infty$.

If we impose more regularity conditions on $\alpha$, we have the following better result.

**Theorem 2.5** Let $\alpha(\cdot)$ and $F$ satisfy the assumptions in Theorem 2.3. Let (2.3) hold. Then for any $u_0 \in H_0^1(0, 1)$, $E(u(t))$ tends to 0 as $t \to \infty$.

**Remark 2.3** We note that the condition (2.3) is almost necessary for stabilization. In fact, by (2.2), it is easy to check that

\[ E(u(t)) = E(u_0) + \int_0^t \int_0^1 \alpha'(x)u^2(s, x)dxds - 2 \int_0^t \int_F u^2(s, x)dxds. \]

Thus if we take $\alpha(x) = 2(1 + x)$ and $F = (0, 1)$, we get $E(u(t)) \equiv E(u(0))$. Thus, the energy does not tend to zero.

**Remark 2.4** Whether the second assertion in Theorem 2.4 holds for any $u_0 \in H_0^1(0, 1)$ or not is an open problem.


3 Sketch of the proof of Theorem 2.2

We denote by $A$ the operator in $L^2(0, 1)$:

\[
\begin{cases}
D(A) \triangleq H^1_0(0, 1) \cap H^2(0, 1), \\
Au = u_{xx}, \quad \forall u \in D(A).
\end{cases}
\]

(3.1)

In order to prove Theorem 2.2, let us re-write (1.2) equivalently as

\[
\begin{cases}
\frac{\partial u}{\partial t} = Au, \quad t > 0, \\
u(0) = u_0,
\end{cases}
\]

where $A : L^2(0, 1; \alpha) \to L^2(0, 1; \alpha)$ is a bounded linear operator given by (recall that $\beta(\cdot) = 0$ and for simplicity we consider only $\alpha(\cdot) \in W(0, 1)$)

\[
Af = (I - A)^{-1} \partial_x (\alpha f), \quad \forall f \in L^2(0, 1; \alpha).
\]

(3.2)

For the proof of Theorem 2.2, it would be useful to have an eigenvector expansion of the solution to (1.2). However, due to the $x$-dependence of $\alpha(x)$, the eigenvectors of $A$ may not be computed explicitly. In fact, we do not know whether the (generalized) eigenvectors of $A$ form a Riesz basis of $L^2(0, 1; \alpha)$ without further regularity conditions on $\alpha$.

In order to overcome the above mentioned difficulty, we need to introduce a special Hilbert space $H$ where the equation evolves by means of a semigroup generated by a compact, skew-adjoint operator. For this purpose, we need some properties of the solutions of (1.2) and (1.3) with $F = (0, a) \cup (b, 1)$. Fix any $\alpha(\cdot) \in W(0, 1)$, and set

\[
\beta_0(x) = e^{1-x}, \quad \gamma_0(x) = e^{x-1}, \\
\beta_n(x) = \int_{x}^{1} \beta_{n-1}(s)(e^{x-s} + e^{s-x})\alpha(s)ds, \\
\gamma_n(x) = \int_{x}^{1} \gamma_{n-1}(s)(e^{x-s} + e^{s-x})\alpha(s)ds,
\]

(3.3)

where $n = 1, 2, \cdots$. We have the following result.

**Lemma 3.1** Suppose that assumptions in Theorem 2.2 hold. Then

\[
\int_{0}^{1} \beta_n(s)\alpha(s)u(t, s)ds = \int_{0}^{1} \gamma_n(s)\alpha(s)u(t, s)ds = 0, \quad \forall t \in (0, T),
\]

where $n = 0, 1, 2, \cdots$.

The Hilbert space $H$ we need is as follows

\[
H \triangleq \left\{ f \in L^2(0, 1; \alpha); \int_{0}^{1} \beta_n(s)\alpha(s)f(s)ds = \int_{0}^{1} \gamma_n(s)\alpha(s)f(s)ds = 0 \right. \\
\left. \text{for } n = 0, 1, 2, \cdots \right\}.
\]

(3.4)

It is easy to see that $H$ is a closed subspace in $L^2(0, 1; \alpha)$. Therefore $H$ is an Hilbert space with the topology inherited from $L^2(0, 1; \alpha)$, i.e. the norm in $H$ is that in $L^2(0, 1; \alpha)$.

We have the following key result.
Lemma 3.2 Let $\alpha(\cdot) \in W(0,1)$ and $A$ and $H$ be defined by (3.2) and (3.4) respectively. Then $A$ is a compact, skew-adjoint operator in $H$.

Lemma 3.2 allows us to use well known spectral theorems to obtain the eigenvector expansion of solutions of (1.2) satisfying (1.3) and to reduce the problem to the unique continuation of the eigenvectors which may be easily solved by ODE techniques. This is sufficient to complete the proof of Theorem 2.2.

4 Sketch of proof of Theorem 2.3

By the time regularity of (1.2), without loss of generality, we may assume $u_0 \in H^1_0(0,1)$. As before, we re-write (1.2) equivalently as

$$
\begin{cases}
u_t = \tilde{A}u, & t > 0, \\
u(0) = u_0,
\end{cases}
$$

where $\tilde{A} : H^1_0(0,1) \to H^1_0(0,1)$ is a bounded linear operator given by (recall (3.1) for $A$

$$
\tilde{A}v = (I - A)^{-1}\left(\partial_x (\alpha v) + \beta v\right), \quad \forall \, v \in H^1_0(0,1).
$$

We need the following simple result.

Lemma 4.1 Let $\alpha(\cdot) \in H^1(0,1)$ and $\beta(\cdot) \in L^2(0,1)$. Then the operator $\tilde{A}$ defined by (4.1) is compact in $H^1(0,1)$, its adjoint operator $\tilde{A}^*$ is given by

$$
\tilde{A}^*w = (I - A)^{-1}\left[-\alpha w_x + \beta w\right], \quad \forall \, w \in H^1_0(0,1).
$$

Furthermore, if $\min_{x \in [0,1]} |\alpha(x)| > 0$, then $0 \notin \sigma_p(\tilde{A}^*)$, the set of eigenvalues of $\tilde{A}^*$.

In order to prove Theorem 2.3, we will show that the generalized eigenvectors of the generator of the underlying semigroup form a Riesz basis of $H^1_0(0,1)$.

We consider only the case $\min_{x \in [0,1]} \alpha(x) > 0$. Put

$$
a_0 \triangleq \int_0^1 \alpha(s)ds, \quad b_0 \triangleq -\frac{1}{a_0} \int_0^1 g(s)ds.
$$

Let us define a linear operator $F : H^1_0(0,1) \to H^1_0(0,1)$ by

$$(Ff)(x) = f \left(\frac{1}{a_0} \int_x^1 \alpha(s)ds\right), \quad \forall \, f \in H^1_0(0,1).$$

It is easy to check that, under the assumptions on $\alpha$ in Theorem 2.3, both $F$ and $F^{-1}$ are bounded linear operators from $H^1_0(0,1)$ to $H^1_0(0,1)$.
Further, denote
\[ u_n(x) = e^{2b_0 x + \frac{1}{b_0} \int_x^1 gds} \frac{\text{sgn}(n)}{\sqrt{1 + n^2 \pi^2}} e^{-i \text{sgn}(n) \sqrt{1 + n^2 \pi^2} x} \sin(n \pi x), \quad n = \pm 1, \pm 2, \ldots. \]

By Proposition 2.3 in [8], it is easy to show that \( \{ \mathcal{F}^{-1} u_n \}_{|n|=1} \) forms a Riesz basis of \( H^1_0(0,1) \).

By means of a careful spectral analysis, we can show the following result.

**Theorem 4.1** Let \( \beta(\cdot) \in L^\infty(0,1) \) and \( \alpha(\cdot) \in W^{2,\infty}(0,1) \) with \( \min_{x \in [0,1]} |\alpha(x)| > 0 \). Then there is a positive integer \( N_1 \) such that for any \( |n| > N_1 \), \( \tilde{\mathcal{A}} \) has a simple eigenvalue \( \mu_n \) with the following asymptotic expansion
\[
\mu_n = \frac{\alpha_0}{2(-b_0 + in\pi)} + O(|n|^{-3});
\]

furthermore, the corresponding eigenvectors \( \phi_n \) satisfy
\[
\sum_{|n|=N_1+1}^{\infty} |\phi_n - \mathcal{F}^{-1} u_n|^2_{H^1_0(0,1)} < \infty.
\]

Theorem 4.1 means that the “high frequency” eigenvectors of \( \tilde{\mathcal{A}} \) are quadratically close to a subsequence of some known Riesz basis in \( H^1_0(0,1) \). In order to prove the completeness of the generalized eigenvectors is also required. But the existing results, for instance those in [5] (that have been successfully applied in several other problems, see, for example, [2], [4], [10] and so on), do not seem to apply in our case. In order to overcome this difficulty, we will need the following new abstract result on Riesz basis property of the generalized eigenvectors of compact operators, which is strongly inspired by Guo’s work ([6]).

**Theorem 4.2** Let \( H \) be an infinite dimensional complex Hilbert space. Let \( G \) be a compact operator in \( H \) and \( 0 \notin \sigma_p(G^*) \). Let \( \{ f_n \}_{n=1}^{\infty} \) be a Riesz basis of \( H \). Suppose a sequence of generalized eigenvectors \( \{ g_n \}_{n=N+1}^{\infty} \) of \( G \) satisfies
\[
\sum_{n=N+1}^{\infty} |g_n - f_n|^2 < \infty
\]
for some \( N \in \mathbb{N} \). Then there exist an integer \( M \geq N \) and some generalized eigenvectors \( \{ g_{n0} \}_{n=1}^{M} \) of \( G \) such that the sequence
\[
\{ g_{n0} \}_{n=1}^{M} \cup \{ g_n \}_{n=M+1}^{\infty}
\]
forms a Riesz basis of \( H \).

Now, combining Theorems 4.1-4.2 and Lemma 4.1, it is easy to arrive at the following conclusion.
Theorem 4.3 Let $\beta(\cdot) \in L^\infty(0,1)$ and $\alpha(\cdot) \in W^{2,\infty}(0,1)$ with $\min_{x \in [0,1]} |\alpha(x)| > 0$. Then there exist finitely many generalized eigenvectors $u_1, u_2, \ldots, u_M$ of $\tilde{A}$ corresponding respectively to eigenvalues $\mu_1, \mu_2, \ldots, \mu_M$ for some integer $M \geq N_1$ such that

$$\{u_k\}_{k=1}^M \cup \{\phi_n\}_{n=M+1}^\infty$$

forms a Riesz basis of $H^1_0(0,1)$, where $N_1$ and $\phi_n$ are given in Theorem 4.1.

Finally, from Theorem 4.3, it is not difficult to complete the proof of Theorem 2.3 by reducing the problem to the unique continuation of the eigenvectors of $\tilde{A}$.

5 Final comments and open problems

First of all, in Theorems 2.2 and 2.3 we impose some technical conditions on $\alpha(\cdot)$, $\beta(\cdot)$, and $F$, especially we require $\alpha(\cdot)$ to be of constant sign. We remark that, in the proof of Theorem 2.2, we use in an essential way the facts that $u$ vanishes in a neighborhood of both extremes $x = 0$ and $x = 1$ (see Lemma 3.1) and $\beta(\cdot) = 0$ (see Lemma 3.2), and we need the constant sign condition on $\alpha(\cdot)$ to check that $\|\cdot\|_{2,\alpha}$ defined by (2.1) (or $\|\cdot\|_{2,-\alpha}$ in the case that $\alpha(\cdot)$ is negative) is a norm in $C_0^\infty(0,1)$. On the other hand, in the proof of Theorem 2.3, we need the constant sign condition on $\alpha(\cdot)$, too (see Lemma 4.1), and we use the fact that $\alpha(\cdot) \in W^{2,\infty}(0,1)$ when we analyze the asymptotic behavior of the eigenvalues and eigenvectors of the generator of the underlying semigroup. However, it is reasonable to expect the (necessary) conditions on $\alpha(\cdot)$, $\beta(\cdot)$ and $F$ in Theorem 2.1 to be also sufficient for the unique continuation property to hold but this is by now an open problem.

Next, it would be interesting to analyze the unique continuation property of the linearized Benjamin-Bona-Mahony equation with potentials depending on both time and space variables. In this case, of course, one can not use the spectral approach developed in this paper. As we remarked before, Holmgren and Carleman methods do not apply to this problem, either (even for the space-dependence potentials). Therefore, it seems that this is really a challenging open problem.

Finally, it would be very interesting to consider the same unique continuation problem but for the true nonlinear Benjamin-Bona-Mahony equation (1.1). The usual linearization method would yield equations with time dependent coefficients in which the techniques of this paper do not apply. However, as our main results (Theorems 2.1–2.3) show, the unique continuation property (even the special case (1.2)) is sensitive to the zero set of the perturbation potentials. Thus, in order to study the unique continuation property of (1.1), one has to consider the special nonlinear structure therein. One possibility would be to use the time analyticity of solutions and suitable power series expansions. But this remains to be done.
References


